

This electronic thesis or dissertation has been downloaded from the King's Research Portal at <https://kclpure.kcl.ac.uk/portal/>



Global and Local Aspects of Supersymmetric Anti-de Sitter Spaces

Beck, Samuel Walker

Awarding institution:
King's College London

The copyright of this thesis rests with the author and no quotation from it or information derived from it may be published without proper acknowledgement.

END USER LICENCE AGREEMENT



Unless another licence is stated on the immediately following page this work is licensed

under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International

licence. <https://creativecommons.org/licenses/by-nc-nd/4.0/>

You are free to copy, distribute and transmit the work

Under the following conditions:

- Attribution: You must attribute the work in the manner specified by the author (but not in any way that suggests that they endorse you or your use of the work).
- Non Commercial: You may not use this work for commercial purposes.
- No Derivative Works - You may not alter, transform, or build upon this work.

Any of these conditions can be waived if you receive permission from the author. Your fair dealings and other rights are in no way affected by the above.

Take down policy

If you believe that this document breaches copyright please contact librarypure@kcl.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.

Global and Local Aspects of Supersymmetric Anti-de Sitter Spaces

Samuel W. Beck

A thesis presented for the degree of
Doctor of Philosophy

Department of Mathematics
Kings College London
United Kingdom
October 19, 2016

Abstract

This dissertation determines the numbers of supersymmetries preserved by the most general warped flux AdS and flat backgrounds in IIA, IIB, and heterotic supergravities. A local analysis determines that $AdS_n \times_w M^{10-n}$ backgrounds preserve $N = 2^{\lfloor \frac{n}{2} \rfloor} k$ supersymmetries for $n \leq 4$ and $N = 2^{\lfloor \frac{n}{2} \rfloor + 1} k$ supersymmetries for $4 < n$. Another local analysis demonstrates that $\mathbb{R}^{1,n-1} \times_w M^{10-n}$ backgrounds preserve $N = 2^{\lfloor \frac{n}{2} \rfloor} k$ supersymmetries for $2 < n \leq 4$, $N = 2^{\lfloor \frac{n+1}{2} \rfloor} k$ supersymmetries for $4 < n \leq 8$, and $N 2^{\lfloor \frac{n}{2} \rfloor} k$ for $n = 9, 10$.

The global analyses show that, with appropriate restrictions, each $AdS_n \times_w M^{10-n}$ background satisfies a Lichnerowicz-type theorem, which generalizes the original Lichnerowicz theorem and proves that the Killing spinors are exactly the zero modes of a Dirac-like operator on M^{10-n} .

Finally, this dissertation includes a non-existence theorem for smooth AdS_5 backgrounds, in 10- or 11-dimensional supergravities, with connected, compact without boundary transverse spaces, that preserve exactly 24 supersymmetries. Any such IIB backgrounds which preserve at least 24 supersymmetries are shown to be locally isometric to $AdS_5 \times S^5$, and any such backgrounds in IIA or 11-dimensional supergravity are shown not to exist.

Acknowledgements

First and foremost I would like to thank my adviser, George Papadopoulos, for his patience and guidance. At every step of my doctoral research, he has helped me to become familiar with these complex concepts and encouraged me to explore these ideas for myself. He has also shown me far more about the ins and outs of academia than I ever could have learned without him. It is only through working with George that I have developed from a student into a researcher, and for that I am eternally grateful.

I would also like to thank my collaborator, Jan Gutowski. Jan's input and insights were crucial to this research. I think it would not have been possible without him.

Finally, I want to thank my family for their support over the years and throughout my education. I am especially grateful to my wife Louise, who has supported me emotionally, has challenged me intellectually, and has made me a better person.

Contents

1	Introduction	7
1.1	Main Results	8
2	AdS Backgrounds	10
2.1	AdS Geometry	10
2.2	Warped Product Spaces	12
2.3	AdS/CFT Correspondence	14
3	IIB Backgrounds	17
3.1	AdS and near horizon geometries	17
3.2	AdS_2 : Local analysis	22
3.3	AdS_2 : Global analysis	24
3.4	AdS_3 : Local analysis	26
3.5	AdS_3 : Global analysis	29
3.6	AdS_4 : Local analysis	31
3.7	AdS_4 : Global analysis	34
3.8	AdS_5 : Local analysis	36
3.9	AdS_5 : Global analysis	37
3.10	AdS_6 : Local analysis	39
3.11	AdS_6 : Global analysis	41
3.12	AdS_n , for $n \geq 7$	42
3.13	Flat IIB backgrounds	43
3.14	On the factorization of Killing spinors	47
3.15	Summary	49
4	IIA Backgrounds	50
4.1	$AdS_2 \times_w M^8$	50
4.2	$AdS_3 \times_w M^7$	55
4.3	$AdS_4 \times_w \mathcal{M}^6$	60
4.4	$AdS_n \times_w M^{10-n}$, $n \geq 5$	64

4.5	Flux $\mathbb{R}^{n-1,1} \times_w M^{10-n}$ backgrounds	68
4.6	On the factorization of Killing spinors	73
4.7	Summary	74
5	Heterotic Backgrounds	76
5.1	AdS_3 backgrounds with $dH = 0$	76
5.2	AdS_3 backgrounds with $dH \neq 0$	86
5.3	A no-go theorem for AdS_n , $n \geq 4$ and $n = 2$ backgrounds	97
5.4	Summary	98
6	AdS_5 Backgrounds with 24 Supersymmetries	99
6.1	$\text{AdS}_5 \times_w M^6$ Solutions in D=11	100
6.2	$\text{AdS}_5 \times_w M^5$ solutions in (massive) IIA supergravity	103
6.3	$\text{AdS}_5 \times_w M^5$ solutions in IIB supergravity	108
6.4	Summary	113
7	Conclusion	114
A	Form and Spinor Conventions	116
B	Notes and Computations	118
B.1	AdS Geometry	118
B.2	Heterotic AdS_3	120
B.3	Heterotic AdS_n , $n \geq 4$	127
B.4	Heterotic $\mathbb{R}^{1,n-1} \times M^{10-n}$	128
B.5	IIA AdS_3	131
B.6	IIA AdS_4	142
B.7	IIA AdS_n , $n \geq 5$	146
B.8	Common Sector AdS_n , $n \geq 4$	153
B.9	IIB AdS-direction KSEs	158
B.10	IIB AdS Directly From Horizon	169
B.11	Homogeneity implies constant warp factor	174
B.12	IIB AdS_3	174
B.13	IIB AdS_4	179
B.14	IIB AdS_5	184
B.15	IIB AdS_6	189

List of Figures

2.1	A cross-section of the Penrose diagram of AdS_n	12
-----	--	----

List of Tables

2.1	Properties of the four sphere-like spaces.	10
3.1	The number of supersymmetries N of $\text{AdS}_n \times_w M^{10-n}$ backgrounds	43
3.2	The number of supersymmetries N of $\mathbb{R}^{1,1} \times_w M^{10-n}$	47
4.1	The number of supersymmetries N of $\text{AdS}_n \times_w M^{10-n}$ backgrounds	69
4.2	The number of supersymmetries N of $\mathbb{R}^{1,1} \times_w M^{10-n}$	73
7.1	SUSY Fraction of AdS_n Backgrounds	114
7.2	SUSY Fraction of $\mathbb{R}^{1,n-1}$ Backgrounds	114

Chapter 1

Introduction

Supersymmetry has long been of interest as one way that the Poincaré algebra can be extended, bypassing the Coleman-Mandula theorem. This theorem states that, under certain assumptions, all internal symmetries must commute with the translation and Lorentz symmetries. A supersymmetry algebra, or superalgebra, violates those assumptions by including fermionic symmetries, which are related to bosonic symmetries by anti-commutation relations [1]. On its own, supersymmetry is promising as a possible extension to the Standard Model, and it is possible that experiments at the Large Hadron Collider will discover superpartners to Standard Model particles in the next few years. For string theory, supersymmetry is a critical component, as superstring theory allows for the fermionic states that bosonic string theory lacks [2].

Theories with both supersymmetry and gravity, in the form of general relativity, are particularly interesting. Because general relativity involves gauged Poincaré symmetry and supersymmetry interacts with Poincaré symmetry non-trivially, the combination implies that supersymmetry must be gauged as well. Because supersymmetry transformations are generated by spinors, the gauge field for supersymmetry is spin- $\frac{3}{2}$, and is part of the same supermultiplet as the spin-2 graviton. This particle is called the "gravitino". In fact, any theory of gauged supersymmetry is necessarily a supergravity theory for the same reason, if supersymmetry is gauged, the Poincaré symmetry must be gauged as well, which implies gravity.

Supergravity also naturally arises as the low-energy limit of string theory. In particular, the spectrum of string modes can be shown to include a massless spin-2 particle, a graviton, and so the low energy limit of string theory includes gravity. If the string theory is supersymmetric, then the resulting field theory is as well, leading to supergravity. It can be shown that a superstring theory must be exactly 10-dimensional to be quantum-mechanically consistent, as only in 10 dimensions do the anomalies cancel out. There are five 10-dimensional string theories, type I, types IIA and IIB, and heterotic $SO(32)$ and $E_8 \times E_8$, and each of these yields a different 10-dimensional supergravity. If these theories are accurate, then our 4-dimensional physics must be the result of Kaluza-Klein dimensional reduction on some compact space. Anti-de Sitter (AdS)

spaces have been used in this context for many years [3, 4].

11-dimensional supergravity, on the other hand, is something of a special case. From a supersymmetry perspective, any space larger than 11 dimensions results in every supermultiplet containing a particle of spin- $\frac{5}{2}$ or greater. Moreover, there is exactly one 11-dimensional supergravity theory [5]. It even turns out that this theory is closely related to string theory, despite all superstring theories being 10-dimensional. It was discovered that all of the string theories and supergravities are related to one another by a variety of dualities, and that IIA supergravity in particular is precisely the Kaluza-Klein dimensional reduction of 11-dimensional supergravity with all higher-order modes omitted. This leads to the idea of 11-dimensional M-theory, which unifies all of these. Like string theory, M-theory involves extended objects, but these objects are only M2-branes and M5-branes, with no fundamental strings. 11-dimensional supergravity turns out to be precisely the low energy limit of M-theory.

In addition to dimensional reduction, AdS supergravity backgrounds have more recently been of interest because of the AdS/CFT correspondence, which relates each AdS background to a dual conformal field theory (CFT) of one less dimension. The maximally supersymmetric $\text{AdS}_5 \times M^5$ background in IIB supergravity, for example, is dual to maximally supersymmetric four-dimensional conformal field theory [6, 7]. In general, the $SO(2, n-1)$ isometry group of AdS_n corresponds exactly to the conformal group of an $(n-1)$ -dimensional conformal field theory, and the supersymmetries of an AdS_n background similarly correspond to the supersymmetries of the dual conformal field theory.

Although many supersymmetric AdS backgrounds have been investigated since then, most efforts focus on specific backgrounds or classes of backgrounds, without attempting to understand AdS backgrounds more generally. In fact, it is not known what superalgebras many of these backgrounds preserve. The work in this dissertation puts such configurations in a broader context, giving information about the numbers of supersymmetries certain backgrounds *can* preserve, as well as new results regarding the forms of the Killing spinors.

There are some previous results which investigate questions similar to those in this dissertation for other types of backgrounds. It has been proven that near-horizon backgrounds preserve an even number of supersymmetries, and Lichnerowicz-type theorems have been proven for IIA horizons [8, 9], IIB horizons [10, 11], and 11-dimensional horizons [12]. Additionally, similar supersymmetry counting results and Lichnerowicz-type theorems have been found and proven for M-theory AdS backgrounds [13].

1.1 Main Results

This dissertation is based on work which has been published in several papers. [14] examines the allowed supersymmetry fractions of IIB backgrounds, and proves a Lichnerowicz-type theorem for each AdS background. [15] uses similar methods to examine IIA backgrounds, and [16] examines heterotic backgrounds, with and without a closed three-form field strength. Finally,

[17] specifically examines AdS_5 backgrounds which preserve at least 24 supersymmetries, showing that those that exist are locally isometric to maximally supersymmetric backgrounds.

Chapters 5, 4, and 3 focus on heterotic backgrounds, IIA backgrounds, and IIB backgrounds, respectively. For each of these backgrounds, a local analysis has been performed which demonstrates supersymmetry enhancement, as well as a Lichnerowicz-type theorem which proves that the Killing spinors of these backgrounds are exactly the zero-modes of a Dirac-like operator.

Each of these analyses depends crucially on the fact that each background is assumed to be invariant under all of the AdS isometries. This severely restricts the forms that the bosonic fields can take, and simplifies the Killing spinor equations. In fact, the AdS -direction gravitino Killing spinor equations are found to be completely integrable in all cases, aside from a single integrability condition, which behaves like an additional algebraic Killing spinor equation on the transverse space. With the problem reduced to the transverse space, supersymmetry enhancement is discovered by finding Clifford-algebra operators which commute with the transverse Killing spinor equations, such that the image of one Killing spinor under these operators is a different, linearly-independent Killing spinor.

The Lichnerowicz-type theorems are proven by assuming that an arbitrary spinor, χ , is a zero-mode of a Dirac-like operator constructed from the transverse Killing spinor equations. Through significant Clifford algebra computations, we can determine that the Laplacian of the length of χ , $\nabla^2 \|\chi\|^2$, is equal to a positive-definite combination of the Killing spinor equations. With the additional assumption that the transverse space is compact, or at least that it satisfies the conditions of the Hopf maximum principle, we can then conclude that the length of χ is constant and the χ is Killing.

Finally, in chapter 6, a proof is presented demonstrating that $\text{AdS}_5 \times_w M^{D-5}$ backgrounds cannot preserve exactly $N = 24$ supersymmetries if M^{D-5} is compact. For 11-dimensional and IIA backgrounds, this proves that such spaces cannot be more than $\frac{1}{2}$ -BPS. For IIB backgrounds, it proves that any such space which is more than $\frac{1}{2}$ -BPS is in fact maximally supersymmetric, and is locally isomorphic to $\text{AdS}_5 \times S^5$.

Chapter 2

AdS Backgrounds

2.1 AdS Geometry

An anti-de Sitter space, AdS_n , is a maximally symmetric spacetime with constant negative curvature. The spaces which are studied in this dissertation not only include an AdS space, but also preserve these symmetries. It is important, therefore, to understand the AdS geometry fully. Much as a sphere, S^n , a maximally symmetric space with constant positive curvature, can be constructed as a subspace of a Euclidean space, \mathbb{R}^{n+1} , an anti-de Sitter space can be constructed as a subspace of a Minkowski space with two time dimensions, $\mathbb{R}^{2,n-1}$. Specifically, the sphere S^n can be constructed as the locus of points in \mathbb{R}^{n+1} which are a constant distance-squared from the origin, $\ell^2 = \sum(x_a)^2$. Similarly a hyperbolic space, \mathbf{H}^n , a maximally symmetric space with constant negative curvature, can be constructed as a subspace of a Minkowski spacetime, $\mathbb{R}^{1,n}$. If the metric signature of the Minkowski space is taken to be $(-, +, +, \dots)$, then \mathbf{H}^n is the locus of points with constant time-like distance squared, $-\ell^2 = -t^2 + \sum(x_a)^2 < 0$. The constructions of de Sitter space (maximally symmetric spacetime with constant positive curvature) and anti-de Sitter space are analogous. A de Sitter space, dS_n , can be constructed as the locus of points in $\mathbb{R}^{1,n}$ with constant space-like distance squared, $\ell^2 = -t^2 + \sum(x_a)^2 > 0$, and an anti-de Sitter space, AdS_n , can be constructed as the locus of points in $\mathbb{R}^{2,n-1}$ with constant time-like distance squared, $-\ell^2 = -(t_1)^2 - (t_2)^2 + \sum(x_a)^2 < 0$.

Space	Signature	Curvature	Subspace of	defined by
S^n	$(+, +, \dots)$	$R > 0$	\mathbb{R}^{n+1}	$\ell^2 = \sum(x_a)^2 > 0$
\mathbf{H}^n	$(+, +, \dots)$	$R < 0$	$\mathbb{R}^{1,n}$	$-\ell^2 = -t^2 + \sum(x_a)^2 < 0$
dS_n	$(-, +, \dots)$	$R > 0$	$\mathbb{R}^{1,n}$	$\ell^2 = -t^2 + \sum(x_a)^2 > 0$
AdS_n	$(-, +, \dots)$	$R < 0$	$\mathbb{R}^{2,n-1}$	$-\ell^2 = -(t_1)^2 - (t_2)^2 + \sum(x_a)^2 < 0$

Table 2.1: Properties of the four sphere-like spaces.

An AdS_n space can equivalently be identified with the coset space $\text{SO}(2, n-1)/\text{SO}(1, n-1)$. To see this, we notice that an $\text{SO}(1, n-1)$ matrix,

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \Lambda & \\ 0 & & & \end{pmatrix},$$

leaves the top row of any $\text{SO}(2, n-1)$ matrix unchanged. We can therefore identify an element of the coset space with the point in $\mathbb{R}^{2, n-1}$ corresponding to its the top row of each matrix, $(t_1, t_2, x_1, \dots, x_{n-1})$. Then, because the matrix is orthogonal, these points must satisfy the condition

$$t_1^2 + t_2^2 - \Sigma(x_a)^2 = 1 ,$$

which defines an AdS space of radius $\ell = 1$. We can identify this with an AdS space of arbitrary radius by scaling the map into $\mathbb{R}^{2, n-1}$ appropriately.

We can determine the AdS metric from the $\mathbb{R}^{2, n-1}$ metric by expressing it in spherical coordinates,

$$ds^2 = -dt_1^2 - dt_2^2 + d\rho^2 + \rho^2 d\Omega_{n-2}^2 .$$

We can define the embedding by

$$\begin{aligned} t_1 &= \ell \sqrt{r^2 + 1} \cos t \\ t_2 &= \ell \sqrt{r^2 + 1} \sin t \\ \rho &= \ell r , \end{aligned}$$

from which we find the metric

$$ds^2 = \ell^2 \left[-(r^2 + 1)dt^2 + \frac{1}{r^2 + 1}dr^2 + r^2 d\Omega_{n-1}^2 \right] . \quad (2.1.1)$$

This spherically symmetric metric is particularly useful because it covers all of AdS space in one coordinate patch, but the Poincaré patch metric will be more useful for the work in this dissertation. We can derive this metric from the embedding in $\mathbb{R}^{2, n-1}$,

$$\begin{aligned} t_1 &= \frac{1}{2z} (\ell^2 + z^2 + \delta_{ab} \hat{x}^a \hat{x}^b - t^2) , \\ t_2 &= \frac{\ell t}{z} , \\ x_a &= \frac{\ell}{z} \hat{x}_a , \quad a = 1, \dots, n-2 , \\ x_{n-1} &= \frac{1}{2z} (-\ell^2 + z^2 + \delta_{ab} \hat{x}^a \hat{x}^b - t^2) , \end{aligned}$$

from which we find the metric,

$$ds^2 = \frac{\ell^2}{z^2} (-dt^2 + dz^2 + \delta_{ab} dx^a dx^b) ,$$

This is analogous to the Poincaré half-plane model of hyperbolic space, which has metric $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$.

To understand the causal structure of an anti-de Sitter space, we want to construct a Penrose diagram for it. To do so, we will start with the global coordinates metric, (2.1.1), using a new radial coordinate, $x = \tan^{-1}(r)$,

$$ds^2 = \ell^2 \sec^2(x) [-dt^2 + dx^2 + \sin^2(x) d\Omega_{n-1}^2] .$$

Then, since x has finite extent, while t has infinite extent, the Penrose diagram is an infinite cylinder [18].

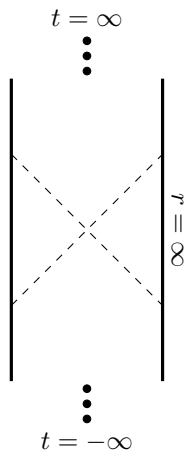


Figure 2.1: A cross-section of the Penrose diagram of AdS_n , including the boundary at $r = \infty$ (thick) and two lightlike geodesics (dashed). Note that the future lightcone of any point reaches $r = \infty$ after finite time, and so $t = \pm\infty$ cannot be shown at finite distance.

2.2 Warped Product Spaces

The anti-de Sitter backgrounds I'll be discussing in this dissertation are warped product spaces, $\text{AdS}_n \times_w M^{D-n}$, which preserve the $\text{SO}(2, n-1)$ isometry group of AdS_n , where $D = 10$ or 11 is the dimension of the supergravity, and M^{D-n} is a transverse space, which is not necessarily compact. Topologically, one of these spaces is an ordinary product space, $\text{AdS}_n \times M^{D-n}$, but the metric is modified by a warp factor, A , which is a function of the transverse space, M^{D-n} . The warped product metric is

$$ds^2 = A^2 ds^2(\text{AdS}_n) + ds^2(M^{D-n}) . \quad (2.2.1)$$

We wish to find coordinates for such a warped product space which fit the near-horizon space of a black hole, because we expect, based on previous results [19, 20, 21, 10, 11, 8, 9, 22, 12], that

near-horizon coordinates will be well-suited to studying the supersymmetry properties of these backgrounds. The near-horizon metric is

$$ds^2 = 2du(dr + rh - \frac{1}{2}r^2\Delta du) + ds^2(M^{D-2}) , \quad (2.2.2)$$

where M^{D-2} is an arbitrary space which will include $n-2$ of the AdS dimensions, and h and Δ are a 1-form and a scalar, respectively, on M^{D-2} .

Starting with the Poincaré patch version of the warped product metric,

$$ds^2 = A^2 \frac{\ell^2}{\hat{z}^2} (-dt^2 + d\hat{z}^2 + \delta_{st} dx^s dx^t) + g_{ij} dy^i dy^j$$

we introduce a new z -coordinate, defined by $\hat{z} = \ell e^{-z/\ell}$, so that the metric becomes

$$ds^2 = A^2 e^{2z/\ell} (-dt^2 + \delta_{ab} dx^a dx^b) + A^2 dz^2 + g_{ij} dy^i dy^j .$$

Then, if $n \geq 3$, we introduce lightcone coordinates $\hat{u} = \frac{1}{\sqrt{2}}(x_1 + t)$ and $\hat{r} = \frac{1}{\sqrt{2}}(x_1 - t)$, so that

$$ds^2 = A^2 e^{2z/\ell} (2d\hat{u}d\hat{r} + \delta_{ab} dx^a dx^b) + A^2 dz^2 + g_{ij} dy^i dy^j .$$

Finally, we introduce rescaled lightcone coordinates, $u = \hat{u}$ and $r = A^2 e^{2z/\ell} \hat{r}$, yielding the final form of the metric,

$$ds^2 = 2du(dr - 2\ell^{-1}rdz - 2rd\ln A) + A^2 dz^2 + A^2 e^{2z/\ell} \delta_{ab} dx^a dx^b + g_{ij} dy^i dy^j . \quad (2.2.3)$$

Comparing this to (2.2.2), we see that it has the same form, with

$$\begin{aligned} h &= -2\ell^{-1}dz - 2d\ln A , \\ \Delta &= 0 , \\ ds^2(M^{D-2}) &= A^2 dz^2 + A^2 e^{2z/\ell} \delta_{ab} dx^a dx^b + g_{ij} dy^i dy^j . \end{aligned}$$

If we're considering an AdS_2 background, however, there is no x_1 coordinate. Instead, we will define u and r as

$$\begin{aligned} u &= 2\sqrt{2}\ell(t - z) , \\ r &= \frac{1}{2\sqrt{2}}\ell A^2 z^{-1} , \end{aligned}$$

so that the metric is

$$ds^2 = 2du\left(dr - 2rd\ln A - \frac{1}{2}\ell^{-2}r^2 A^{-2}du\right) + g_{ij} dy^i dy^j . \quad (2.2.4)$$

Comparing this to (2.2.2), we see that it has the same form, with

$$\begin{aligned} h &= -2d\ln A , \\ \Delta &= \ell^{-2}A^{-2} , \\ ds^2(M^{D-2}) &= g_{ij} dy^i dy^j . \end{aligned}$$

We will also need vielbein forms for each of these spaces. The forms we will use for AdS_2 spaces are

$$\begin{aligned}\mathbf{e}^+ &= du , \\ \mathbf{e}^- &= dr - 2rd \ln A - \frac{1}{2}\ell^{-2}r^2 A^{-2} du ,\end{aligned}$$

so that the AdS_2 metric, (2.2.4), is $ds^2 = 2\mathbf{e}^+\mathbf{e}^- + g_{ij}y^i y^j$. The forms we will use for AdS_n spaces, $n \geq 3$, are

$$\begin{aligned}\mathbf{e}^+ &= du , \\ \mathbf{e}^- &= dr - 2\ell^{-1}rdz - 2rd \ln A , \\ \mathbf{e}^z &= Adz , \\ \mathbf{e}^a &= Ae^{z/\ell} dx^a ,\end{aligned}$$

so that the AdS_n metric, (2.2.3), is $ds^2 = 2\mathbf{e}^+\mathbf{e}^- + (\mathbf{e}^z)^2 + \delta_{ab}\mathbf{e}^a\mathbf{e}^b + g_{ij}dy^i dy^j$.

One of the primary advantages of these lightcone vielbein forms is that we can construct projection operators from the corresponding Gamma matrices, Γ_+ and Γ_- . Specifically, we can write any spinor as a unique sum, $\epsilon = \epsilon_+ + \epsilon_-$, where $\Gamma_{\pm}\epsilon_{\pm} = 0$.

It will be necessary, for the Lichnerowicz-type theorems, to assume that the transverse spaces are compact, but this assumption will not be necessary for the other parts of this dissertation.

The fields of these backgrounds are also assumed to be invariant under the Killing vectors of the AdS space. This restricts the form of each field. Most of the fields of M-theory and heterotic, IIA, and IIB supergravities are differential forms. A k -form F which is invariant under the AdS Killing vectors of $\text{AdS}_n \times_w M^{D-n}$ will have the form

$$F = X \wedge d\text{vol}(\text{AdS}_n) + Y , \quad \text{if } k \geq n ,$$

or

$$F = Y , \quad \text{if } k < n ,$$

where X is an $(n - k)$ -form on M^{D-n} and Y is a k -form on M^{D-n} .

2.3 AdS/CFT Correspondence

The major motivation for the work in this dissertation is the AdS/CFT correspondence, which relates string theory on an $n + 1$ -dimensional background to a conformal field theory on the n -dimensional boundary of that space. The best understood example of this is the $\text{AdS}_5/\text{CFT}_4$ correspondence, which starts with the observation that a IIB supergravity solution consisting of N D3-branes becomes $\text{AdS}_5 \times S^5$ in the near-horizon limit.

2.3.1 D3 Branes

Consider a IIB supergravity solution consisting of N coincident D3-branes. The Dirac-Born-Infeld action for these branes is found to be a generalization of the Yang-Mills action, with coupling constant

$$g_{\text{YM}}^2 = 2\pi g_s . \quad (2.3.1)$$

The metric of this configuration, which preserves the Poincaré symmetry of the four dimensions parallel to the branes and the $SO(6)$ rotational symmetry of the transverse dimensions, is

$$ds^2 = H(r)^{-\frac{1}{2}} \eta_{\mu\nu} dx^\mu dx^\nu + H(r)^{\frac{1}{2}} \delta_{ij} dy^i dy^j , \quad (2.3.2)$$

where x^μ are the parallel dimensions, y^i are the transverse dimensions, and r is the radial coordinate defined by $r^2 = y_i y^i$. $H(r)$ is a harmonic function on the transverse coordinates, $H(r) = 1 + \frac{L^4}{r^4}$.

The constant L is determined by the self-dual five-form, given by

$$F = -H(r)^{-2} H'(r) (dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dr + *_{10} (dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dr)) .$$

By integrating this over a 5-sphere centered on the D3-brane we can determine the net D3-brane charge, $Q = \frac{1}{4\pi g_s \alpha'^2} L^4$, and setting it equal to N , we find that

$$L^4 = 4\pi g_s N \alpha'^2 . \quad (2.3.3)$$

The D3-brane metric has two relevant limits, $r \gg L$ and $r \ll L$. When $r \gg L$, $H(r) \sim 1$, and we recover 10-dimensional Minkowski space, $\mathbb{R}^{1,9}$. On the other hand, when $r \ll L$, $H(r) \sim \frac{L^4}{r^4}$. If we rewrite the transverse metric in spherical coordinates, $\delta_{ij} dy^i dy^j = dr^2 + r^2 d\Omega_5^2$, then the D3-brane metric becomes

$$ds^2 = \frac{r^2}{L^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{L^2}{r^2} dr^2 + L^2 d\Omega_5^2 .$$

Introducing $z = \frac{L^2}{r}$, this finally becomes

$$ds^2 = \frac{L^2}{z^2} (\eta_{\mu\nu} dx^\mu dx^\nu + dz^2) + L^2 d\Omega_5^2 , \quad (2.3.4)$$

which we recognize as $\text{AdS}_5 \times S^5$, with L as both the AdS radius and the S^5 radius.

2.3.2 Supergravity Limit

The specific relationship between the string theory on the AdS space and the dual $SU(N)$ Yang-Mills theory is summarized by the relations (2.3.1) and (2.3.3), which can be rewritten

$$g_{\text{YM}}^2 = 2\pi g_s , \quad 2g_{\text{YM}}^2 N = \frac{L^4}{\alpha'^2} . \quad (2.3.5)$$

The corresponding supergravity background is the limit of this system in which the string coupling, g_s , is small, and the string length, $\ell_s = \sqrt{\alpha'}$ is small compared to the AdS radius, L ,

i.e. the limit in which $g_s \rightarrow 0$ and $\frac{\alpha'}{L^2} \rightarrow 0$. On the Yang-Mills side, this is equivalent to the limit in which $g_{\text{YM}}^2 \rightarrow 0$, but $g_{\text{YM}}^2 N \equiv \lambda \rightarrow \infty$. The great value, then, in studying supergravity AdS backgrounds in particular, is that they are dual to Yang-Mills theories with large 't Hooft coupling, which are otherwise typically difficult to analyze.

2.3.3 Dual Fields

We can use Kaluza-Klein dimensional reduction to reduce IIB supergravity to the AdS₅ directions. For each supergravity field φ , suppressing the ten-dimensional spacetime indices, we can write φ in terms of spherical harmonics on S^5 , $Y_{\ell,I}(\Omega_5)$, which satisfy $\nabla_{S^5}^2 Y_{\ell,I} = -L^{-2}\ell(\ell+4)Y_{\ell,I}$, as

$$\varphi = \sum_{\ell=0}^{\infty} \sum_{I_{\ell}} \varphi_{\ell,I_{\ell}}(x^{\mu}, z) Y_{\ell,I_{\ell}}(\Omega^5) .$$

In general, the non-zero components of $\varphi_{\ell,I_{\ell}}$ with S^5 are restricted by the supergravity field equations, but they will still take this general form.

To understand the CFT duals of these components, we will consider the representations of SO(6) formed by the spherical harmonics. If S^5 is embedded in \mathbb{R}^6 , then the spherical harmonics can be written in terms of the \mathbb{R}^6 coordinates as

$$Y_{\ell,I} = C_{i_1, \dots, i_{\ell}}^{\ell, I_{\ell}} x^{i_1} \cdots x^{i_{\ell}} ,$$

where $C_{i_1, \dots, i_{\ell}}^{\ell, I_{\ell}}$ is a traceless symmetric tensor. These form the same representation of SO(6) as the $\frac{1}{2}$ -BPS CFT operators \mathcal{O}^{Δ} of scaling dimension $\Delta = \ell$.

2.3.4 General Dimension

The arguments above cannot be replicated for a general AdS _{n} supergravity background, but it is still possible to motivate an AdS _{n} /CFT _{$n-1$} correspondence. In specific cases, particularly the AdS₄ \times S^7 and AdS₇ \times S^4 M-theory backgrounds, the arguments are similar. More generally, if we notice that the isometry group of AdS _{n} is the same as the conformal group of CFT _{$n-1$} , SO(2, $n-1$), we can relate AdS fields to CFT operators by their transformations under these symmetries.

Chapter 3

IIB Backgrounds

IIB AdS backgrounds have been of particular interest since the original AdS/CFT duality relates a IIB $\text{AdS}_5 \times S^5$ background to a super-Yang Mills theory [6]. Many backgrounds of this type have been found in the years since [23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. Even so, to date there has been no general analysis of the numbers of supersymmetries these backgrounds preserve.

In this chapter, a local analysis is developed which demonstrates supersymmetry enhancement for all IIB AdS backgrounds. AdS_n backgrounds are found to always have the same numbers of supersymmetries as IIA AdS_n backgrounds, $N = 2^{\lfloor \frac{n}{2} \rfloor} k$ supersymmetries for $2 \leq n \leq 4$ and $N = 2^{\lfloor \frac{n}{2} \rfloor + 1} k$ supersymmetries for $5 \leq n \leq 7$, where $k \in \mathbb{Z}$.

Additionally, a Lichnerowicz-type theorem is proven for each AdS background discussed. The original Lichnerowicz theorem tells us that, for any space with zero scalar curvature, all of the zeroes of the Dirac operator are parallel spinors. Similarly, with these theorems we find that all of the zeroes of a Dirac-like operator constructed from the Killing spinor equations are actually Killing spinors. In this case, the scalar component of the Einstein equation plays the role of the flatness condition.

3.1 AdS and near horizon geometries

3.1.1 Warped AdS and flat backgrounds

The warped AdS and flat backgrounds can be written universally as near horizon geometries [19]. Let F , G and P be the 5-, 3- and 1-form field strengths of IIB supergravity. All AdS backgrounds can be described in terms of the fields

$$\begin{aligned} ds^2 &= 2\mathbf{e}^+ \mathbf{e}^- + ds^2(\mathcal{S}) , & F &= r\mathbf{e}^+ \wedge X + \mathbf{e}^+ \wedge \mathbf{e}^- \wedge Y + \star_8 Y , \\ G &= r\mathbf{e}^+ \wedge L + \mathbf{e}^+ \wedge \mathbf{e}^- \wedge \Phi + H , & P &= \xi , \end{aligned} \tag{3.1.1}$$

where we have introduced the frame

$$\mathbf{e}^+ = du, \quad \mathbf{e}^- = dr + rh - \frac{1}{2}r^2 \Delta du, \quad \mathbf{e}^i = e_I^i dy^I , \tag{3.1.2}$$

and

$$ds^2(\mathcal{S}) = \delta_{ij} \mathbf{e}^i \mathbf{e}^j \quad (3.1.3)$$

is the metric on the horizon spatial section \mathcal{S} which is co-dimension 2 submanifold given by the equations $r = u = 0$. In addition, the self-duality of F requires that $X = -\star_8 X$. The dependence on the coordinates u and r is explicitly given. Δ , h , Y are 0-, 1- and 3-forms on \mathcal{S} , respectively, Φ , L and H are λ -twisted 1-, 2- and 3-forms on \mathcal{S} , respectively, and ξ is a λ^2 -twisted 1-form on \mathcal{S} , where λ arises from the pull back of the canonical bundle on the scalar manifold¹ $SU(1,1)/U(1)$ on \mathcal{S} . Furthermore, the Bianchi identities imply that

$$X = d_h Y - \frac{i}{8}(\Phi \wedge \bar{H} - \bar{\Phi} \wedge H), \quad L = d_h \Phi - i\Lambda \wedge \Phi + \xi \wedge \bar{\Phi}. \quad (3.1.4)$$

and so X and L are not independent fields.

Moreover, viewing the backgrounds $AdS_n \times_w M^{10-n}$ as a near horizon geometries, the spatial horizon sections \mathcal{S} are $\mathcal{S} = H^{n-2} \times_w M^{10-n}$, ie warped products of hyperbolic (n-2)-dimensional space with M^{10-n} . This can be easily seen after the fields are stated explicitly for each case below.

Although all AdS backgrounds are described by (3.1.1), the field dependence of individual AdS cases differs. To address this, we shall separately state the fields in each case as follows.

$AdS_2 \times_w M^8$

In this case $M^8 = \mathcal{S}$ and the fields become

$$\begin{aligned} ds^2 &= 2du(dr + rh - \frac{1}{2}r^2\Delta du) + ds^2(M^9), \quad F = \mathbf{e}^+ \wedge \mathbf{e}^- \wedge Y + \star_8 Y, \\ G &= \mathbf{e}^+ \wedge \mathbf{e}^- \wedge \Phi + H, \quad P = \xi, \end{aligned} \quad (3.1.5)$$

where

$$h = -2A^{-1}dA = \Delta^{-1}d\Delta, \quad X = L = 0, \quad (3.1.6)$$

Observe that $dh = 0$.

$AdS_3 \times_w M^7$

The fields are

$$\begin{aligned} ds^2 &= 2du(dr + rh) + A^2 dz^2 + ds^2(M^7), \quad F = A\mathbf{e}^+ \wedge \mathbf{e}^- \wedge dz \wedge Y - \star_7 Y \\ G &= A\mathbf{e}^+ \wedge \mathbf{e}^- \wedge dz \wedge \Phi + H, \quad P = \xi, \end{aligned} \quad (3.1.7)$$

where

$$h = -\frac{2}{\ell}dz - 2A^{-1}dA, \quad \Delta = 0, \quad X = L = 0, \quad (3.1.8)$$

and ℓ is the radius of AdS .

¹The scalar manifold can also be taken as the fundamental domain of the modular group but we shall not dwell on this.

$AdS_4 \times_w M^6$

The field are

$$\begin{aligned} ds^2 &= 2du(dr + rh) + A^2(dz^2 + e^{2z/\ell} dx^2) + ds^2(M^7), \quad F = A^2 e^{z/\ell} \mathbf{e}^+ \wedge \mathbf{e}^- \wedge dz \wedge dx \wedge Y + \star_6 Y \\ G &= H, \quad P = \xi, \end{aligned} \quad (3.1.9)$$

where

$$h = -\frac{2}{\ell} dz - 2A^{-1} dA, \quad \Delta = 0, \quad X = L = 0. \quad (3.1.10)$$

$AdS_5 \times_w M^5$

The fields are

$$\begin{aligned} ds^2 &= 2du(dr + rh) + A^2(dz^2 + e^{2z/\ell}(dx^2 + dy^2) + ds^2(M^5)), \quad G = H, \quad P = \xi, \\ F &= Y \left[A^3 e^{2z/\ell} \mathbf{e}^+ \wedge \mathbf{e}^- \wedge dz \wedge dx \wedge dy - \text{dvol}(M^5) \right], \end{aligned} \quad (3.1.11)$$

where

$$h = -\frac{2}{\ell} dz - 2A^{-1} dA, \quad \Delta = 0, \quad X = L = 0. \quad (3.1.12)$$

$AdS_6 \times_w M^4$

The fields are

$$\begin{aligned} ds^2 &= 2du(dr + rh) + A^2(dz^2 + e^{2z/\ell} \left(\sum_{a=1}^3 (dx^a)^2 \right) + ds^2(M^4)), \quad F = 0, \\ G &= H, \quad P = \xi, \end{aligned} \quad (3.1.13)$$

where

$$h = -\frac{2}{\ell} dz - 2A^{-1} dA, \quad \Delta = 0, \quad X = L = 0. \quad (3.1.14)$$

It should be noted that the warped backgrounds $\mathbb{R}^{n-1,1} \times_w M^{10-n}$ are included in our analysis. They arise in the limit that the AdS radius ℓ goes to infinity. This limit is smooth for all our field configurations presented above. However, some statements that apply for AdS do not extend to the flat backgrounds. Because of this some care must be taken when adapting the results we obtain for AdS backgrounds to the limit of infinite radius.

3.1.2 Bianchi identities and field equations

It is clear from the expressions of the fields for the AdS backgrounds in the previous section that $L = X = 0$ and $dh = 0$. As a result, we have

$$d_h Y - \frac{i}{8} (\Phi \wedge \bar{H} - \bar{\Phi} \wedge H) = 0, \quad d_h \Phi - i\Lambda \wedge \Phi + \xi \wedge \bar{\Phi} = 0. \quad (3.1.15)$$

Furthermore, the remaining Bianchi identities for the backgrounds (3.1.1) are

$$\begin{aligned} d \star_8 Y &= \frac{i}{8} H \wedge \bar{H} , \quad dH = i\Lambda \wedge H - \xi \wedge \bar{H} , \\ d\xi &= 2i\Lambda \wedge \xi , \quad d\Lambda = -i\xi \wedge \bar{\xi} , \end{aligned} \quad (3.1.16)$$

where Λ is a $U(1)$ connection of λ , see [11] for more details.

The independent field equations of the AdS backgrounds (3.1.1) are

$$\tilde{\nabla}^i \Phi_i - i\Lambda^i \Phi_i - \xi^i \bar{\Phi}_i + \frac{2i}{3} Y_{\ell_1 \ell_2 \ell_3} H^{\ell_1 \ell_2 \ell_3} = 0 , \quad (3.1.17)$$

$$\tilde{\nabla}^\ell H_{\ell ij} - i\Lambda^\ell H_{\ell ij} - h^\ell H_{\ell ij} - \xi^\ell \bar{H}_{\ell ij} + \frac{2i}{3} (\star_8 Y_{ij \ell_1 \ell_2 \ell_3} H^{\ell_1 \ell_2 \ell_3} - 6Y_{ij \ell} \Phi^\ell) = 0 , \quad (3.1.18)$$

$$\tilde{\nabla}^i \xi_i - 2i\Lambda^i \xi_i - h^i \xi_i + \frac{1}{24} (-6\Phi^2 + H^2) = 0 , \quad (3.1.19)$$

$$\frac{1}{2} \tilde{\nabla}^i h_i - \Delta - \frac{1}{2} h^2 + \frac{2}{3} Y^2 + \frac{3}{8} \Phi^i \bar{\Phi}_i + \frac{1}{48} \|H\|^2 = 0 , \quad (3.1.20)$$

and

$$\begin{aligned} \tilde{R}_{ij} + \tilde{\nabla}_{(i} h_{j)} - \frac{1}{2} h_i h_j + 4Y_{ij}^2 + \frac{1}{2} \Phi_{(i} \bar{\Phi}_{j)} - 2\xi_{(i} \bar{\xi}_{j)} - \frac{1}{4} H_{\ell_1 \ell_2 (i} \bar{H}_{j)}^{\ell_1 \ell_2} \\ + \delta_{ij} \left(-\frac{1}{8} \Phi_\ell \bar{\Phi}^\ell - \frac{2}{3} Y^2 + \frac{1}{48} \|H\|^2 \right) = 0 , \end{aligned} \quad (3.1.21)$$

where \tilde{R} is the Ricci tensor of \mathcal{S} . There is an additional field equation which is not independent because they follow from those above. This is

$$\frac{1}{2} \tilde{\nabla}^2 \Delta - \frac{3}{2} h^i \tilde{\nabla}_i \Delta - \frac{1}{2} \Delta \tilde{\nabla}^i h_i + \Delta h^2 = 0 , \quad (3.1.22)$$

which we state because it is useful in the investigation of the KSEs.

3.1.3 Killing spinor equations

The gravitino and dilatino KSEs of IIB supergravity [33, 34] are

$$\left(\nabla_M - \frac{i}{2} Q_M + \frac{i}{48} \not{F}_M \right) \epsilon - \frac{1}{96} \left(\Gamma \not{G}_M - 9 \not{G}_M \right) C * \epsilon = 0 , \quad (3.1.23)$$

$$\not{P} C * \epsilon + \frac{1}{24} \not{G} \epsilon = 0 , \quad (3.1.24)$$

respectively, where Q is a $U(1)$ connection of λ .

These KSEs can be solved for the fields (3.1.1) along the directions u, r . For this first decompose $\epsilon = \epsilon_+ + \epsilon_-$, where $\Gamma_{\pm}\epsilon_{\pm} = 0$. Then a direct substitution into the (3.1.23) and (3.1.24) reveals that the Killing spinor can be expressed as

$$\epsilon_+ = \phi_+, \quad \epsilon_- = \phi_- + r\Gamma_- \Theta_+ \phi_+; \quad \phi_+ = \eta_+ + u\Gamma_+ \Theta_- \eta_-, \quad \phi_- = \eta_-, \quad (3.1.25)$$

where η_{\pm} do not depend on both u and r coordinates and

$$\Theta_{\pm} = \left(\frac{1}{4} \not{h} \pm \frac{i}{12} \not{Y} \right) + \left(\frac{1}{96} \not{\#} \pm \frac{3}{16} \not{\Phi} \right) C * . \quad (3.1.26)$$

After some extensive computation using the field equations described in [11], one can show that the independent KSEs for the backgrounds (3.1.1) are

$$\nabla_i^{(\pm)} \eta_{\pm} = 0, \quad \mathcal{A}^{(\pm)} \eta_{\pm} = 0, \quad (3.1.27)$$

where

$$\begin{aligned} \nabla_i^{(\pm)} &\equiv \tilde{\nabla}_i + \left(-\frac{i}{2} \Lambda_i \mp \frac{1}{4} h_i \mp \frac{i}{4} Y_i \pm \frac{i}{12} \Gamma Y_i \right) \\ &\quad + \left(\pm \frac{1}{16} \Gamma \not{\Phi}_i \mp \frac{3}{16} \not{\Phi}_i - \frac{1}{96} \Gamma \not{\#}_i + \frac{3}{32} \not{\#}_i \right) C * , \end{aligned} \quad (3.1.28)$$

and

$$\mathcal{A}^{(\pm)} \equiv \left(\mp \frac{1}{4} \not{\Phi} + \frac{1}{24} \not{\#} \right) + \not{\xi} C * . \quad (3.1.29)$$

It turns out that (3.1.27) are the restriction of the (3.1.23) and (3.1.24) on the horizon section \mathcal{S} for ϵ given in (3.1.25).

Furthermore, one can show that if η_- is a solution to the KSEs, then

$$\eta_+ = \Gamma_+ \Theta_- \eta_- \quad (3.1.30)$$

also solves the KSEs. This is the first indication that IIB horizons exhibit supersymmetry enhancement. Indeed if \mathcal{S} is compact and the fluxes do not vanish, one can show [11] that $\text{Ker} \Theta_- = \{0\}$ and so η_+ given in the above equation yields an additional supersymmetry.

Although the following integrability conditions

$$\left(\frac{1}{2} \Delta + 2\Theta_- \Theta_+ \right) \eta_+ = 0, \quad \left(\frac{1}{2} \Delta + 2\Theta_+ \Theta_- \right) \eta_- = 0, \quad (3.1.31)$$

are implied from the above KSEs, it is convenient for the analysis that follows to include them. As we shall see, they are instrumental in the solution of the KSEs along the AdS_n directions for $n > 2$.

3.1.4 Horizon Dirac equations

Before we complete this section, we shall summarize the results of [11] on the relation between Killing spinors and zero modes of Dirac-like operators for IIB horizons. We have seen that the gravitino KSE gives rise to two parallel transport equations on \mathcal{S} associated with the covariant derivatives $\nabla^{(\pm)}$ (3.2.12). If S^\pm are the complex chiral spin bundles over \mathcal{S} , then $\nabla^\pm : \Gamma(S^\pm \otimes \lambda^{\frac{1}{2}}) \rightarrow \Gamma(\Lambda^1(\mathcal{S}) \otimes S^\pm \otimes \lambda^{\frac{1}{2}})$, where $\Gamma(S^\pm \otimes \lambda^{\frac{1}{2}})$ are the smooth sections of $S^\pm \otimes \lambda^{\frac{1}{2}}$. In turn, one can define the associated horizon Dirac operators

$$\mathcal{D}^{(\pm)} \equiv \Gamma^i \nabla_i^{(\pm)} = \Gamma^i \tilde{\nabla}_i + \Psi^\pm, \quad (3.1.32)$$

where

$$\Psi^\pm \equiv \Gamma^i \Psi_i^{(\pm)} = -\frac{i}{2} \not{A} \mp \frac{1}{4} \not{Y} \pm \frac{i}{6} \not{Y} + \left(\pm \frac{1}{4} \not{\Phi} + \frac{1}{24} \not{H} \right) C * \quad (3.1.33)$$

Clearly the ∇^\pm parallel spinors are zero modes of $\mathcal{D}^{(\pm)}$. For \mathcal{S} compact, one can also prove the converse, ie that all zero modes of the horizon Dirac equations $\mathcal{D}^{(\pm)}$ are Killing spinors. Therefore, one can establish

$$\nabla^{(\pm)} \eta_\pm = 0, \quad \mathcal{A}^{(\pm)} \eta_\pm = 0 \iff \mathcal{D}^{(\pm)} \eta_\pm = 0. \quad (3.1.34)$$

The proof of the above statement for η_+ spinors utilizes the Hopf maximum principle on $\|\eta_+\|^2$ while for η_- employs a partial integration formula. In the former case, one also finds that $\|\eta_+\| = \text{const.}$ Similar theorems have been proven for other theories in [22, 12].

3.2 AdS_2 : Local analysis

3.2.1 Fields, Bianchi identities and field equations

For AdS_2 backgrounds $M^8 = \mathcal{S}$ and the fields on \mathcal{S} are

$$ds^2(\mathcal{S}) = ds^2(M^9), \quad \tilde{F}^3 = Y, \quad \tilde{F}^5 = \star_8 Y, \quad \tilde{G}^1 = \Phi, \quad \tilde{G}^3 = H, \quad \tilde{P} = \xi \quad (3.2.1)$$

Next, we set

$$\Delta = \ell^{-2} A^{-2}, \quad (3.2.2)$$

which satisfies (3.1.22), where ℓ is the radius of AdS_2 . Using these, the Bianchi identities (3.1.15) and (3.1.16) can now be written as

$$d(A^{-2}Y) - \frac{iA^{-2}}{8}(\Phi \wedge \bar{H} - \bar{\Phi} \wedge H) = 0, \quad d(A^{-2}\Phi) - iA^{-2}\Lambda \wedge \Phi + A^{-2}\xi \wedge \bar{\Phi} = 0. \quad (3.2.3)$$

$$d\star_8 Y = \frac{i}{8}H \wedge \bar{H}, \quad dH = i\Lambda \wedge H - \xi \wedge \bar{H},$$

$$d\xi = 2i\Lambda \wedge \xi, \quad d\Lambda = -i\xi \wedge \bar{\xi}, \quad (3.2.4)$$

respectively, where Λ is a $U(1)$ connection of λ restricted on \mathcal{S} .

Similarly, the field equations read as

$$\tilde{\nabla}^i \Phi_i - i\Lambda^i \Phi_i - \xi^i \bar{\Phi}_i + \frac{2i}{3} Y_{\ell_1 \ell_2 \ell_3} H^{\ell_1 \ell_2 \ell_3} = 0, \quad (3.2.5)$$

$$\tilde{\nabla}^\ell H_{\ell ij} - i\Lambda^\ell H_{\ell ij} + 2A^{-1} \partial^\ell A H_{\ell ij} - \xi^\ell \bar{H}_{\ell ij} + \frac{2i}{3} (\star_8 Y_{ij \ell_1 \ell_2 \ell_3} H^{\ell_1 \ell_2 \ell_3} - 6Y_{ij \ell} \Phi^\ell) = 0, \quad (3.2.6)$$

$$\tilde{\nabla}^i \xi_i - 2i\Lambda^i \xi_i + 2A^{-1} \partial^i A \xi_i + \frac{1}{24} (-6\Phi^2 + H^2) = 0, \quad (3.2.7)$$

$$-A^{-1} \tilde{\nabla}^2 A - A^{-2} \partial^i A \partial_i A - \ell^{-2} A^{-2} + \frac{2}{3} Y^2 + \frac{3}{8} \Phi^i \bar{\Phi}_i + \frac{1}{48} \|H\|^2 = 0, \quad (3.2.8)$$

and

$$\begin{aligned} R_{ij}^{(8)} - 2A^{-1} \tilde{\nabla}_i \partial_j A + 4Y_{ij}^2 + \frac{1}{2} \Phi_{(i} \bar{\Phi}_{j)} - 2\xi_{(i} \bar{\xi}_{j)} - \frac{1}{4} H_{\ell_1 \ell_2 (i} \bar{H}_{j)}^{\ell_1 \ell_2} \\ + \delta_{ij} \left(-\frac{1}{8} \Phi_\ell \bar{\Phi}^\ell - \frac{2}{3} Y^2 + \frac{1}{48} \|H\|^2 \right) = 0, \end{aligned} \quad (3.2.9)$$

where $R^{(8)}$ is the Ricci tensor of $\mathcal{S} = M^8$.

The warped factor A is no-where vanishing

To see this, assume that A is not identically zero. Thus there is a point in M^8 such that $A \neq 0$. Multiplying (3.2.8) with A^2 evaluated as a point for which $A \neq 0$, one finds

$$-A \tilde{\nabla}^2 A - \partial^i A \partial_i A - \ell^{-2} + \frac{2}{3} A^2 Y^2 + \frac{3}{8} A^2 \Phi^i \bar{\Phi}_i + \frac{1}{48} A^2 \|H\|^2 = 0, \quad (3.2.10)$$

Next taking a sequence that converges at a point where A vanishes, one finds an inconsistency as the term involving the AdS radius ℓ cannot vanish. Therefore there are no smooth solutions for which A vanishes at some point on the spacetime. A more detailed argument for this has been presented in [13].

This property depends crucially on ℓ taking a finite value. In particular, it is not valid in the limit that ℓ goes to infinity, and so one cannot conclude that A is no-where vanishing for flat backgrounds.

3.2.2 Killing spinor equations

The KSEs on $\mathcal{S} = \mathcal{M}^8$ are

$$\nabla_i^{(\pm)} \eta_{\pm} = 0, \quad \mathcal{A}^{(\pm)} \eta_{\pm} = 0, \quad (3.2.11)$$

where

$$\begin{aligned} \nabla_i^{(\pm)} &\equiv \tilde{\nabla}_i + \left(-\frac{i}{2} \Lambda_i \pm \frac{1}{2} A^{-1} \partial_i A \mp \frac{i}{4} \mathcal{Y}_i \pm \frac{i}{12} \Gamma \mathcal{Y}_i \right) \\ &\quad + \left(\pm \frac{1}{16} \Gamma \Phi_i \mp \frac{3}{16} \Phi_i - \frac{1}{96} \Gamma \mathcal{H}_i + \frac{3}{32} \mathcal{H}_i \right) C * , \end{aligned} \quad (3.2.12)$$

and

$$\mathcal{A}^{(\pm)} \equiv \left(\mp \frac{1}{4} \Phi + \frac{1}{24} \mathcal{H} \right) + \xi C * . \quad (3.2.13)$$

Furthermore, if η_- is a Killing spinor, then

$$\eta_+ = \Gamma_+ \Theta_- \eta_- \quad (3.2.14)$$

is also a Killing spinor, where now

$$\Theta_{\pm} = \left(-\frac{1}{4} \not{\partial} \log A^2 \pm \frac{i}{12} \mathcal{Y} \right) + \left(\frac{1}{96} \mathcal{H} \pm \frac{3}{16} \Phi \right) C * . \quad (3.2.15)$$

It is not apparent that $\eta_+ \neq 0$ as η_- may be in the Kernel of Θ_- . To establish under which conditions $\eta_+ \neq 0$, one has to impose additional restrictions on M^8 . However if $\eta_+ \neq 0$, then the solutions exhibit supersymmetry enhancement.

3.2.3 Counting supersymmetries

The analysis so far is not sufficient to establish either the formulae regarding the number of supersymmetries N preserved by the AdS_2 backgrounds. For this, some additional restrictions on M^8 are required. We shall explore these in the next section.

3.3 AdS_2 : Global analysis

The main results of this section are to demonstrate that under certain assumptions, there is a 1-1 correspondence between Killing spinors and zero modes of Dirac operators on M^8 coupled to fluxes, and use this to count the supersymmetries N of AdS_2 backgrounds. Given the gravitino KSE in (3.2.11) and in particular the (super)covariant derivatives $\nabla^{(\pm)}$, one can construct the Dirac-like operators

$$\mathcal{D}^{(\pm)} \equiv \Gamma^i \nabla_i^{(\pm)} = \Gamma^i \tilde{\nabla}_i + \Psi^{\pm} , \quad (3.3.1)$$

on M^8 , where

$$\Psi^\pm \equiv \Gamma^i \Psi_i^{(\pm)} = -\frac{i}{2} \mathbb{A} \pm \frac{1}{4} \not{\partial} \log A^2 \pm \frac{i}{6} \mathbb{Y} + \left(\pm \frac{1}{4} \not{\Phi} + \frac{1}{24} \not{\mathbb{H}} \right) C * \quad (3.3.2)$$

Clearly all parallel spinors η_\pm , ie $\nabla^{(\pm)} \eta_\pm = 0$, are zero modes of $\mathcal{D}^{(\pm)}$, ie $\mathcal{D}^{(\pm)} \eta_\pm = 0$. The task is to prove the converse.

3.3.1 A Lichnerowicz theorem for $\mathcal{D}^{(+)}$

The proof of this converse is a Lichnerowicz type of theorem and the proof is similar as that given in [11] for horizon Dirac operators. Because of this, we shall not give details of the proof. The novelty of this theorem is that the converse implies that the zero modes of $\mathcal{D}^{(+)}$ solve both the gravitino and dilatino KSEs. In particular, assuming that $\mathcal{D}^{(+)} \eta_+ = 0$ and after some algebra which involves the use of field equations, one can establish that

$$\tilde{\nabla}^i \tilde{\nabla}_i \|\eta_+\|^2 + \partial^i \log A^2 \tilde{\nabla}_i \|\eta_+\|^2 = 2 \|\nabla^{(+)} \eta_+\|^2 + \|\mathcal{A}^{(+)} \eta_+\|^2. \quad (3.3.3)$$

It is then a consequence of the maximum principle that the only solution of the above equation is $\|\eta_+\| = \text{const}$ and that η_+ is a Killing spinor. In particular, this is the case provided M^8 is compact.

3.3.2 A Lichnerowicz theorem for $\mathcal{D}^{(-)}$

The proof the zero modes of the $\mathcal{D}^{(-)}$ are Killing spinors is similar to that for the $\mathcal{D}^{(+)}$ operator. In particular, if $\mathcal{D}^{(-)} \eta_- = 0$, then one can show that

$$\tilde{\nabla}^i \tilde{\nabla}_i \|\eta_-\|^2 + h^i \tilde{\nabla}_i \|\eta_-\|^2 + \tilde{\nabla}^i h_i \|\eta_-\|^2 = 2 \|\nabla^{(-)} \eta_-\|^2 + \|\mathcal{A}^{(-)} \eta_-\|^2. \quad (3.3.4)$$

Using $h = d \log \Delta$, this can be rewritten as

$$\tilde{\nabla}^i \tilde{\nabla}_i (\Delta \|\eta_-\|^2) - h^i \tilde{\nabla}_i (\Delta \|\eta_-\|^2) = 2\Delta \|\nabla^{(-)} \eta_-\|^2 + \Delta \|\mathcal{A}^{(-)} \eta_-\|^2. \quad (3.3.5)$$

The maximum principle again implies that the only solutions to this equation are those for which $\Delta \|\eta_-\| = \text{const}$ and η_- are Killing spinors. Again this is always the case if M^8 is compact. It should be noted that unlike the case of general IIB horizons where this theorem has been proven using a partial integration formula [11], here we have presented a different proof based on the maximum principle. The latter has an advantage as it gives some additional information regarding the length of the Killing spinor η_- . Combining the results of this section with those of the previous one, we have established that if M^8 is compact, then

$$\nabla^{(\pm)} \eta_\pm = 0, \quad \mathcal{A}^{(\pm)} \eta_\pm = 0 \iff \mathcal{D}^{(\pm)} \eta_\pm = 0, \quad (3.3.6)$$

and that

$$\|\eta_+\| = \text{const}, \quad \Delta \|\eta_-\| = \text{const}. \quad (3.3.7)$$

3.3.3 Counting supersymmetries again

The number of supersymmetries of AdS_2 backgrounds is

$$N = N_- + N_+ \quad (3.3.8)$$

where

$$N_{\pm} = \dim \text{Ker}(\nabla^{(\pm)}, \mathcal{A}^{(\pm)}) . \quad (3.3.9)$$

Using the correspondence between the Killing spinors and zero modes of the $\mathcal{D}^{(\pm)}$ operators in (3.3.6), we conclude that

$$N = \dim \text{Ker} \mathcal{D}^{(-)} + \dim \text{Ker} \mathcal{D}^{(+)} . \quad (3.3.10)$$

As for near horizon geometries [11], one can prove that $\dim \text{Ker} \mathcal{D}^{(+)\dagger} = \dim \text{Ker} \mathcal{D}^{(-)}$. This is done by a direct observation upon comparing the adjoint of $\mathcal{D}^{(+)}$ with $\mathcal{D}^{(-)}$. As a result for M^8 compact without boundary, we find that

$$N = \text{Index}(\mathcal{D}^{(+)}) + 2\dim \text{Ker} \mathcal{D}^{(+)} = 2(N_- + \text{Index}(D)) \quad (3.3.11)$$

where D is the Dirac operator twisted with $\lambda^{\frac{1}{2}}$. The index of $\mathcal{D}^{(+)}$ is twice the index of D because they have the same principal symbol and $\mathcal{D}^{(+)}$ acts on two copies of the Majorana-Weyl representation of M^8 . This establishes that $N = 2k$ for AdS_2 backgrounds.

Furthermore, if M^8 is compact without boundary with a η_- Killing spinor, one can explicitly construct a η_+ Killing spinor by setting $\eta_+ = \Gamma_+ \Theta_- \eta_-$. This is because if M^8 is compact without boundary and the fluxes do not vanish, then $\text{Ker} \Theta_- = \{0\}$. The proof of this statement is similar to that demonstrated in [11] for near horizon geometries and so it will not be repeated here.

We have shown that the number of supersymmetries preserved by AdS_2 backgrounds is even. Apart from this, there are additional restrictions on N . In particular, it has been shown in [35, 36] but if a IIB background preserves more than 28 supersymmetries, $N > 28$, then it is maximally supersymmetric. Moreover, the maximal supersymmetric and the solutions preserving 28 supersymmetries have been classified in [37] and [38], respectively, and they do not include AdS_2 backgrounds. From these, one concludes that $N \leq 26$. One can also adapt the proof of [39] to this case to demonstrate that all AdS_2 backgrounds preserving more than 16 supersymmetries are homogeneous. This in particular implies that the IIB scalars are constant for all these backgrounds.

3.4 AdS_3 : Local analysis

3.4.1 Fields, Bianchi identities and field equations

The fields restricted on the spatial horizon section $\mathcal{S} = \mathbb{R} \times_w M^7$ are

$$ds^2(\mathcal{S}) = A^2 dz^2 + ds^2(M^7) , \quad \tilde{F}^3 = Adz \wedge Y , \quad \tilde{F}^5 = -\star_7 Y$$

$$\tilde{G}^1 = A\Phi dz, \quad \tilde{G}^3 = H, \quad \tilde{P}^1 = \xi. \quad (3.4.1)$$

Moreover, we have that $h = -\frac{2}{\ell}dz - 2A^{-1}dA$ and $\Delta = X = L = 0$.

Substituting these into the Bianchi identities (3.1.15) and (3.1.16), we find that

$$dY = -3d\log A \wedge Y + \frac{i}{8}(\bar{\Phi}H - \Phi\bar{H}) \quad (3.4.2)$$

$$d\Phi = 3\Phi d\log A + i\Phi Q - \bar{\Phi}\xi \quad (3.4.3)$$

$$d *_7 Y = -\frac{i}{8}H \wedge \bar{H} \quad (3.4.4)$$

$$dH = iQ \wedge H - \xi \wedge \bar{H} \quad (3.4.5)$$

$$d\xi = 2iQ \wedge \xi \quad (3.4.6)$$

$$dQ = -i\xi \wedge \bar{\xi}. \quad (3.4.7)$$

In addition the field equations (3.1.17)-(3.1.21) give

$$\nabla^i H_{ijk} = -3A^{-1}\partial^i A H_{ijk} + iQ^i G_{ijk} + P^i H_{ijk} + 4i\Phi Y_{jk} + \frac{i}{3}\epsilon_{jki_1 i_2 \ell_1 \ell_2 \ell_3} Y^{i_1 i_2} H^{\ell_1 \ell_2 \ell_3} \quad (3.4.8)$$

$$\nabla^i \xi_i = -3A^{-1}\partial^i A \xi_i + 2iQ^i \xi_i - \frac{1}{24}H^2 - \frac{1}{4}\Phi^2 \quad (3.4.9)$$

$$A^{-1}\nabla^2 A = 2Y^2 + \frac{3}{8}\|\Phi\|^2 + \frac{1}{48}\|H\|^2 - 2\ell^{-2}A^{-2} - 2(d\log A)^2 \quad (3.4.10)$$

$$R_{ij}^{(7)} = 2A^{-1}\nabla_i \nabla_j A + 2Y^2 \delta_{ij} - 8Y_{ij}^2 + \frac{1}{4}H_{(i}{}^{k\ell}\bar{H}_{j)k\ell} + \frac{1}{8}\|\Phi\|^2 \delta_{ij} - \frac{1}{48}\|H\|^2 \delta_{ij} + 2\xi_{(i}\bar{\xi}_{j)}, \quad (3.4.11)$$

where $R^{(7)}$ is the Ricci tensor of M^7 . Contracting this, we find that

$$R^{(7)} = \frac{3}{A}\nabla^2 A + 6Y^2 + \frac{5}{48}\|H\|^2 + \frac{7}{8}\|\Phi\|^2 + 2|\xi|^2 \quad (3.4.12)$$

$$= -\frac{6}{\ell^2}A^{-2} - 6(A^{-1}dA)^2 + 12Y^2 + 2\|\Phi\|^2 + \frac{1}{6}\|H\|^2 + 2\|\xi\|^2. \quad (3.4.13)$$

3.4.2 The warped factor is no-where vanishing

One of the consequence of the field equations is that the warped factor A is no-where vanishing. One can show that this follows from the field equation (3.4.10) using an argument similar to that presented for the AdS_2 backgrounds.

3.4.3 Solution of Killing spinor equations

To integrate the KSEs along the AdS_3 directions, it suffices to integrate the horizon KSEs (3.1.27) along the z coordinate. For this consider first the gravitino KSE. Evaluating the expression along the z -coordinate, we find

$$\partial_z \eta_{\pm} = \Xi_{\pm} \eta_{\pm} \quad (3.4.14)$$

where

$$\Xi_{\pm} = \mp \frac{1}{2\ell} - \frac{1}{2}\Gamma_z \not{A} \pm \frac{i}{4}A\gamma + \left(\frac{1}{96}A\Gamma_z \not{H} \pm \frac{3}{16}A\Phi \right) C * . \quad (3.4.15)$$

Observe that

$$\Xi_+ = A\Gamma_z \Theta_+ , \quad \Xi_- = \frac{1}{\ell} + A\Gamma_z \Theta_- . \quad (3.4.16)$$

Next differentiating (3.4.14) and comparing the resulting expression with the integrability conditions (3.1.31), one finds that

$$\partial_z^2 \eta_{\pm} \pm \frac{1}{\ell} \partial_z \eta_{\pm} = 0 , \quad (3.4.17)$$

which can be solved to give

$$\eta_{\pm} = \sigma_{\pm} + e^{\mp \frac{z}{\ell}} \tau_{\pm} , \quad (3.4.18)$$

where

$$\Xi_{\pm} \sigma_{\pm} = 0 , \quad \Xi_{\pm} \tau_{\pm} = \mp \frac{1}{\ell} \tau_{\pm} , \quad (3.4.19)$$

with both σ_{\pm} and τ_{\pm} z -independent spinors. The latter conditions are additional independent algebraic KSEs.

Although, we have solved along the z direction, there are potentially additional conditions that can arise from mixed integrability conditions along the z -direction and the remaining directions in \mathcal{S} . However, it can be shown after some computation that this is not the case. Furthermore, the dilatino KSEs in (3.1.27) restrict on the σ_{\pm} and τ_{\pm} spinors in a straightforward manner. This completes the integration of the KSEs along all AdS_3 directions. The remaining independent KSEs, which are localized on M^7 , are

$$\begin{aligned} \nabla_i^{(\pm)} \sigma_{\pm} &= 0 , \quad \nabla_i^{(\pm)} \tau_{\pm} = 0 , \\ \mathcal{A}^{(\pm)} \sigma_{\pm} &= 0 , \quad \mathcal{A}^{(\pm)} \tau_{\pm} = 0 , \\ \mathcal{B}^{(\pm)} \sigma_{\pm} &= 0 , \quad \mathcal{C}^{(\pm)} \tau_{\pm} = 0 , \end{aligned} \quad (3.4.20)$$

where

$$\begin{aligned} \nabla_i^{(\pm)} &= \nabla_i + \Psi_i^{(\pm)} , \quad \mathcal{A}^{(\pm)} = \mp \frac{1}{4} \Phi \Gamma_z + \frac{1}{24} \not{H} + \not{g} C * , \\ \mathcal{B}^{(\pm)} &= \Xi_{\pm} , \quad \mathcal{C}^{(\pm)} = \Xi_{\pm} \pm \frac{1}{\ell} \end{aligned} \quad (3.4.21)$$

and

$$\begin{aligned} \Psi_i^{(\pm)} &= \pm \frac{1}{2} \partial_i \log A - \frac{i}{2} Q_i \pm \frac{i}{4} (\Gamma \not{Y})_i \Gamma_z \mp \frac{i}{2} \gamma_i \Gamma^z \\ &\quad + \left(-\frac{1}{96} (\Gamma \not{H})_i + \frac{3}{32} \not{H}_i \mp \frac{1}{16} \Phi \Gamma_{zi} \right) C * . \end{aligned} \quad (3.4.22)$$

Therefore, there are four sets of three independent KSEs on M^7 . Having found a solution to the above equations, one can substitute in (3.4.18) and then in (3.1.25) to find the Killing spinors for the $AdS_3 \times_w M^7$ background.

3.4.4 Counting supersymmetries

It is straightforward to observe that if one has an either σ_- or a τ_- solution, then

$$\sigma_+ = A^{-1}\Gamma_z\Gamma_+\sigma_- , \quad \tau_+ = A^{-1}\Gamma_z\Gamma_+\tau_- , \quad (3.4.23)$$

are also solutions of the independent KSEs (3.4.20). Conversely, if either σ_+ or τ_+ are solutions, then

$$\sigma_- = A\Gamma_z\Gamma_-\sigma_+ , \quad \tau_- = A\Gamma_z\Gamma_-\tau_+ , \quad (3.4.24)$$

are also solutions to the KSEs (3.4.20). Therefore, we have that the number of Killing spinors N of the AdS_3 backgrounds are

$$\begin{aligned} N &= 2(\dim \text{Ker}(\nabla^{(-)}, \mathcal{A}^{(-)}, \mathcal{B}^{(-)}) + \dim \text{Ker}(\nabla^{(-)}, \mathcal{A}^{(-)}, \mathcal{C}^{(-)})) \\ &= 2(\dim \text{Ker}(\nabla^{(+)}, \mathcal{A}^{(+)}, \mathcal{B}^{(+)}) + \dim \text{Ker}(\nabla^{(+)}, \mathcal{A}^{(+)}, \mathcal{C}^{(+)}) \end{aligned} \quad (3.4.25)$$

Thus the AdS_3 backgrounds preserve even number of supersymmetries. This proves the formula for N for AdS_3 backgrounds.

The number of supersymmetries N of AdS_3 backgrounds are further restricted. It follows from the results of [35, 36, 38] that there are no supersymmetric AdS_3 backgrounds preserving more than 28 supersymmetries. As a result, $N \leq 26$.

3.5 AdS_3 : Global analysis

The main task here is to show the formula for counting the number of supersymmetries of AdS_3 backgrounds. For this, we have to show that there is a 1-1 correspondence between Killing spinors and zero modes of a Dirac-like operator on M^7 .

3.5.1 A Lichnerowicz theorem for τ_+ and σ_+

To prove that the zero modes of a Dirac-like operator on M^7 are Killing spinors, one has to determine an appropriate Dirac-like operator on M^7 . The naive Dirac-like operator which one can construct from contracting $\nabla^{(\pm)}$ with a gamma matrix is not suitable. Instead, let us modify the parallel transport operators of the gravitino KSE as

$$\begin{aligned} \check{\nabla}_i^{(+)} &= \nabla_i^{(+)} + q\Gamma_{zi}A^{-1}\mathcal{B}^{(+)} , \\ \hat{\nabla}_i^{(+)} &= \nabla_i^{(+)} + q\Gamma_{zi}A^{-1}\mathcal{C}^{(+)} , \end{aligned} \quad (3.5.1)$$

on σ_+ and τ_+ , respectively, where q is a number which later will be set to $1/7$. It is clear that if either σ_+ or τ_+ are Killing spinors, they are also parallel with respect to the above covariant derivatives.

Since the analysis that follows is similar for σ_+ and τ_+ , it is convenient to present it in a unified way. For this write both (3.5.1) as

$$\mathbb{D}_i^{(+)} = \nabla_i^{(+)} + q\Gamma_{zi}A^{-1}\mathbb{B}^{(+)} \quad (3.5.2)$$

where

$$\mathbb{B}^{(+)} = -\frac{c}{2\ell} - \frac{1}{2}\Gamma_z\partial A + \frac{i}{4}A\dot{Y} + \left(\frac{1}{96}A\Gamma_z\# + \frac{3}{16}A\Phi\right)C* \quad (3.5.3)$$

and $c = 1$ when acting on σ_+ and $c = -1$ when acting on τ_+ , ie either $\mathbb{B}^{(+)} = \mathcal{B}^{(+)}$ or $\mathbb{B}^{(+)} = \mathcal{C}^{(+)}$, respectively.

Next define the Dirac-like operators

$$\mathcal{D}^{(+)} \equiv \Gamma^i\mathbb{D}_i^{(+)} = \Gamma^i\nabla_i + \Sigma^{(+)} . \quad (3.5.4)$$

where

$$\begin{aligned} \Sigma^{(+)} = & \frac{7qc}{2\ell}A^{-1}\Gamma_z + \frac{1+7q}{4}\partial\log A^2 - \frac{i}{2}\mathcal{Q} + \frac{3i-7iq}{4}\dot{Y}\Gamma_z \\ & + \left(\frac{5-7q}{96}\# + \frac{7-21q}{16}\Phi\right)C* . \end{aligned} \quad (3.5.5)$$

It turns that $\mathcal{D}^{(+)}$ is suitable to formulate a maximum principle on the length square of σ_+ and τ_+ . In particular, suppose that χ_+ is a zero mode for $\mathcal{D}^{(+)}$, ie $\mathcal{D}^{(+)}\chi_+ = 0$, where $\chi_+ = \sigma_+$ for $c = 1$ while $\chi_+ = \tau_+$ for $c = -1$. Then after some Clifford algebra, which is presented in appendix B.12, which requires the use of field equations and for $q = 1/7$, one can establish the identity

$$\begin{aligned} \nabla^2\|\chi_+\|^2 + 3A^{-1}\partial^i A\partial_i\|\chi_+\|^2 = & 2\left\|\mathbb{D}^{(+)}\chi_+\right\|^2 \\ & + \frac{16}{7}A^{-2}\left\|\mathbb{B}^{(+)}\chi_+\right\|^2 + \left\|\mathcal{A}^{(+)}\chi_+\right\|^2 . \end{aligned} \quad (3.5.6)$$

Assuming that M^7 satisfies the requirements of the Hopf maximum principle, eg for M^7 compact, the above equation implies that χ_+ is a Killing spinor and that the length $\|\chi_+\| = \text{const}$.

To summarize, we have shown that

$$\begin{aligned} \nabla_i^{(+)}\sigma_+ = 0 , \quad \mathcal{B}^{(+)}\sigma_+ = 0 , \quad \mathcal{A}^{(+)}\sigma_+ = 0 & \iff \mathcal{D}^{(+)}\sigma_+ = 0 ; \quad c = 1 \\ \nabla_i^{(+)}\tau_+ = 0 , \quad \mathcal{C}^{(+)}\tau_+ = 0 , \quad \mathcal{A}^{(+)}\tau_+ = 0 & \iff \mathcal{D}^{(+)}\tau_+ = 0 ; \quad c = -1 , \end{aligned} \quad (3.5.7)$$

and that

$$\|\sigma_+\| = \text{const} , \quad \|\tau_+\| = \text{const} . \quad (3.5.8)$$

3.5.2 A Lichnerowicz theorem for τ_- and σ_-

A similar theorem to that presented in the previous section can be presented for τ_- and σ_- spinors. One can define the operators $\mathbb{D}^{(-)}$ and $\mathcal{D}^{(-)}$ and repeat the analysis. Alternatively,

one can observe that χ_+ is a zero mode of the $\mathcal{D}^{(+)}$ operator, then $\chi_- = A\Gamma_z\chi_+$ is a zero mode of the $\mathcal{D}^{(-)}$ operator, where χ_- is either σ_- or τ_- . Since the same relation holds between χ_+ and χ_- Killing spinors, one can establish a maximum principle for χ_- spinor. The formula is that given in (3.5.6) after setting $\chi_+ = A^{-1}\Gamma_z\chi_-$. Therefore provided the requirements of Hopf maximum principle are satisfied, one establishes

$$\begin{aligned} \nabla_i^{(-)}\sigma_- = 0, \quad \mathcal{B}^{(-)}\sigma_- = 0, \quad \mathcal{A}^{(-)}\sigma_- = 0 &\iff \mathcal{D}^{(-)}\sigma_- = 0; \quad c = 1 \\ \nabla_i^{(-)}\tau_- = 0, \quad \mathcal{C}^{(-)}\tau_- = 0, \quad \mathcal{A}^{(-)}\tau_- = 0 &\iff \mathcal{D}^{(-)}\tau_- = 0; \quad c = -1, \end{aligned} \quad (3.5.9)$$

and that

$$\|\sigma_-\|^2 = A^2 \text{const}, \quad \|\tau_-\|^2 = A^2 \text{const}, \quad (3.5.10)$$

where

$$\mathcal{D}^{(-)} = \Gamma^i \nabla_i + \Sigma^{(-)}, \quad (3.5.11)$$

and

$$\begin{aligned} \Sigma^{(-)} = & -\frac{7qc}{2\ell}A^{-1}\Gamma_z + \frac{-1+7q}{4}\not{D}\log A^2 - \frac{i}{2}\not{Q} - \frac{3i-7iq}{4}\not{Y}\Gamma_z \\ & + \left(\frac{5-7q}{96}\not{H} - \frac{7-21q}{16}\not{\Phi}\right)C*. \end{aligned} \quad (3.5.12)$$

3.5.3 Counting supersymmetries again

The proof of the relation between Killing spinors and the zero modes of the Dirac-like operators $\mathcal{D}^{(\pm)}$ allows us to re-express the number of supersymmetries N in (3.4.25) preserved by AdS_3 backgrounds as

$$\begin{aligned} N &= 2(\dim \text{Ker } \mathcal{D}_{c=1}^{(-)} + \dim \text{Ker } \mathcal{D}_{c=-1}^{(-)}) \\ &= 2(\dim \text{Ker } \mathcal{D}_{c=1}^{(+)} + \dim \text{Ker } \mathcal{D}_{c=-1}^{(+)}), \end{aligned} \quad (3.5.13)$$

which establishes that $N = 2k$ for AdS_3 .

3.6 AdS_4 : Local analysis

3.6.1 Fields, Bianchi identities and field equations

The field on \mathcal{S} are

$$\begin{aligned} ds^2(\mathcal{S}) &= A^2(dz^2 + e^{2z/\ell}dx^2) + ds^2(M^6), \quad \tilde{F}^3 = A^2 e^{z/\ell} dz \wedge dx \wedge Y, \quad \tilde{F}^5 = \star_6 Y \\ \tilde{G}^3 &= H, \quad P = \xi, \end{aligned} \quad (3.6.1)$$

with $h = -\frac{2}{\ell}dz - 2A^{-1}dA$ and $\Delta = X = L = 0$. Substituting these into the Bianchi and field equations on \mathcal{S} in section 3.1.2 reduce on M^6 as follows. The Bianchi identities give

$$d(A^4 Y) = 0, \quad \tilde{\nabla}^i Y_i = -\frac{i}{288} \epsilon^{i_1 i_2 i_3 j_1 j_2 j_3} H_{i_1 i_2 i_3} \bar{H}_{j_1 j_2 j_3} \quad (3.6.2)$$

$$dH = iQ \wedge H - \xi \wedge \bar{H} \quad (3.6.3)$$

$$d\xi = 2iQ \wedge \xi \quad (3.6.4)$$

$$dQ = -i\xi \wedge \bar{\xi}. \quad (3.6.5)$$

Therefore the Binachi identities imply that $A^4 Y$ is a closed 1-form and that $H \wedge \bar{H}$ represents a trivial cohomology class in M^6 .

The Einstein equation on \mathcal{S} gives

$$A^{-1} \nabla^2 A = 4Y^2 + \frac{1}{48} \|H\|^2 - \frac{3}{A^2} - 3(A^{-1} dA)^2, \quad (3.6.6)$$

and

$$\begin{aligned} R_{ij}^{(6)} - 4A^{-1} \nabla_i \nabla_j A - 4Y^2 \delta_{ij} + 8Y_i Y_j \\ - \frac{1}{4} H_{(i}{}^{k\ell} \bar{H}_{j)k\ell} + \frac{1}{48} \|H\|^2 \delta_{ij} - 2\xi_{(i} \bar{\xi}_{j)} = 0, \end{aligned} \quad (3.6.7)$$

where $R^{(6)}$ is the Ricci tensor of M^6 . The remaining field equations are

$$\begin{aligned} \nabla^i H_{ijk} &= -3\partial^i \log A H_{ijk} + iQ^i H_{ijk} + \xi^i \bar{H}_{ijk}, \\ \nabla^i \xi_i &= -3\partial^i \log A \xi_i + 2iQ^i \xi_i - \frac{1}{24} H^2. \end{aligned} \quad (3.6.8)$$

This concludes reduction of the Bianchi identities and field equations on M^6 .

The warped factor is no-where vanishing

One consequence of the field equations and in particular of (3.6.6) is that the warp factor A is no-where vanishing. The investigation for this is similar to that we have presented for AdS_3 and so we shall not repeat the argument here.

3.6.2 Solution of KSEs

The integration of the KSEs along the z -coordinate proceeds as in the AdS_3 . In particular repeating the argument as in the AdS_3 case, one finds that

$$\eta_{\pm} = \phi_{\pm} + e^{\mp z/\ell} \chi_{\pm} \quad (3.6.9)$$

where

$$\Xi_{\pm} \phi_{\pm} = 0, \quad \Xi_{\pm} \chi_{\pm} = \mp \frac{1}{\ell} \chi_{\pm}, \quad \mathcal{A}^{(\pm)} \phi_{\pm} = 0, \quad \mathcal{A}^{(\pm)} \chi_{\pm} = 0 \quad (3.6.10)$$

and

$$\Xi_{\pm} = \mp \frac{1}{2\ell} - \frac{1}{2} \Gamma_z \not{A} \pm \frac{i}{2} A \Gamma_x \not{Y} + \frac{1}{96} A \Gamma_z \not{C} *,$$

$$\mathcal{A}^{(\pm)} = \frac{1}{24} \not{H} + \not{C} * . \quad (3.6.11)$$

Observe that although $\mathcal{A}^{(+)} = \mathcal{A}^{(-)}$ as operators, they act on different spaces and so we shall retain the distinct labeling.

Next we integrate the gravitino KSE along the x AdS coordinate to obtain

$$\eta_+ = \sigma_+ - \frac{1}{\ell} x \Gamma_x \Gamma_z \tau_+ + e^{-z/\ell} \tau_+ , \quad \eta_- = \sigma_- + e^{z/\ell} \left(-\frac{1}{\ell} x \Gamma_x \Gamma_z \sigma_- + \tau_- \right) , \quad (3.6.12)$$

where

$$\Xi_{\pm} \sigma_{\pm} = 0 , \quad \Xi_{\pm} \tau_{\pm} = \mp \frac{1}{\ell} \tau_{\pm} , \quad (3.6.13)$$

and σ_{\pm} and τ_{\pm} depend only on the coordinates of M^6 . This completes the integration of the gravitino KSE along all AdS_4 directions. The dilatino KSE simply restricts on the spinors σ_{\pm} and τ_{\pm} . There are no additional conditions arises from integrability conditions between AdS_4 and M^6 directions.

Therefore, remaining independent KSEs on M^6 are

$$\begin{aligned} \nabla_i^{(\pm)} \sigma_{\pm} &= 0 , \quad \nabla_i^{(\pm)} \tau_{\pm} = 0 , \\ \mathcal{A}^{(\pm)} \sigma_{\pm} &= 0 , \quad \mathcal{A}^{(\pm)} \tau_{\pm} = 0 , \\ \mathcal{B}^{(\pm)} \sigma_{\pm} &= 0 , \quad \mathcal{C}^{(\pm)} \tau_{\pm} = 0 , \end{aligned} \quad (3.6.14)$$

where

$$\nabla_i^{(\pm)} = \nabla_i + \Psi_i^{(\pm)} , \quad \mathcal{B}^{(\pm)} = \Xi_{\pm} , \quad \mathcal{C}^{(\pm)} = \Xi_{\pm} \pm \frac{1}{\ell} , \quad (3.6.15)$$

and

$$\begin{aligned} \Psi_i^{(\pm)} &= \pm \frac{1}{2} \partial_i \log A - \frac{i}{2} Q_i \mp \frac{i}{2} (\Gamma \not{Y})_i \Gamma_{xz} \pm \frac{i}{2} Y_i \Gamma_{xz} \\ &\quad + \left(-\frac{1}{96} (\Gamma \not{H})_i + \frac{3}{32} \not{H}_i \right) C * . \end{aligned} \quad (3.6.16)$$

This concludes the reduction of the KSEs on M^6 .

3.6.3 Counting supersymmetries

As for AdS_3 backgrounds there are Clifford algebra operators which interwind between the different KSEs on M^6 . In particular observe that if σ_{\pm} is a solution to the KSEs, then

$$\tau_{\pm} = \Gamma_z \Gamma_x \sigma_{\pm} \quad (3.6.17)$$

is also a solution, and vice versa. Furthermore as for AdS_3 , if either σ_- or τ_- is a solution, so is

$$\sigma_+ = A^{-1} \Gamma_+ \Gamma_z \sigma_- , \quad \tau_+ = A^{-1} \Gamma_+ \Gamma_z \tau_- . \quad (3.6.18)$$

Similarly, if either σ_+ or τ_+ is a solution, so is

$$\sigma_- = A\Gamma_- \Gamma_z \sigma_+ , \quad \tau_- = A\Gamma_- \Gamma_z \tau_+ . \quad (3.6.19)$$

From the above relations one concludes that the $AdS_4 \times_w M^6$ backgrounds preserve

$$N = 4 \dim \text{Ker}(\nabla^{(\pm)}, \mathcal{A}^{(\pm)}, \mathcal{B}^{(\pm)}) = 4 \dim \text{Ker}(\nabla^{(\pm)}, \mathcal{A}^{(\pm)}, \mathcal{C}^{(\pm)}) , \quad (3.6.20)$$

for either $+$ or $-$ choice of sign. This confirms $N = 4k$ for the AdS_4 backgrounds.

The number of supersymmetries N of AdS_4 backgrounds are further restricted. It is a consequence of [37, 35, 36, 38] that there are no AdS_4 backgrounds with $N \geq 28$ supersymmetries. Therefore $N \leq 24$.

3.7 AdS_4 : Global analysis

3.7.1 A Lichnerowicz theorem for τ_{\pm} and σ_{\pm}

Next, we will demonstrate a Lichnerowicz type of theorem which states there is a 1-1 correspondence between Killing spinors and the zero modes of Dirac-like operators on M^6 coupled to fluxes. The proof is similar to that we have presented for the AdS_3 backgrounds. However, the operators involved in the AdS_4 case are different and so the proof is not a mere repetition.

We shall present the proof of the the Lichnerowicz type of theorem for σ_+ and τ_+ spinors. The proof for the other pair σ_- and τ_- follows as a consequence. It is also convenient to do the computations simultaneously for both σ_+ and τ_+ spinors which from now on we shall call collectively χ_+ .

To begin let us define the operator

$$\mathbb{D}_i^{(+)} = \nabla_i^{(+)} + q\Gamma_{zi}A^{-1}\mathbb{B}^{(+)} \quad (3.7.1)$$

where

$$\mathbb{B}^{(+)} = -\frac{c}{2\ell} - \frac{1}{2}\Gamma_z \not{A} - \frac{i}{2}A \not{Y} \Gamma_x + \frac{1}{96}A\Gamma_z \not{H} C^* \quad (3.7.2)$$

and $c = 1$ when acting on σ_+ and $c = -1$ when acting on τ_+ , ie either $\mathbb{B}^{(+)} = \mathcal{B}^{(+)}$ or $\mathbb{B}^{(+)} = \mathcal{C}^{(+)}$, respectively. It is clear from this that if χ_+ is a Killing spinor, then it is parallel with respect to \mathbb{D} .

Next define the Dirac-like operator

$$\mathcal{D}^{(+)} \equiv \Gamma^i \mathbb{D}_i^{(+)} = \Gamma^i \nabla_i + \Sigma^{(+)} . \quad (3.7.3)$$

where

$$\Sigma^{(+)} = \frac{3qc}{2}A^{-1}\Gamma_z + \frac{1+6q}{4}\not{A} \log A^2 - \frac{i}{2}\not{Q} + (2i - 3iq)\not{Y} \Gamma_{zx} + \frac{1-q}{16}\not{H} C^* . \quad (3.7.4)$$

Next suppose that χ_+ is a zero mode of $\mathcal{D}^{(+)}$, ie $\mathcal{D}^{(+)}\chi_+ = 0$. Then after some Clifford algebra computation, which has been presented in appendix B.13, $q = 1/3$, and the use of field equations, one can establish the identity

$$\begin{aligned} \nabla^2 \|\chi_+\|^2 + 4A^{-1} \partial^i A \partial_i \|\chi_+\|^2 &= 2 \left\| \mathbb{D}^{(+)} \chi_+ \right\|^2 \\ &+ \frac{16}{3} A^{-2} \left\| \mathbb{B}^{(+)} \chi_+ \right\|^2 + \left\| \mathcal{A}^{(+)} \chi_+ \right\|^2 . \end{aligned} \quad (3.7.5)$$

Assuming that requirements of the Hopf maximum principle are satisfied, eg for M^6 compact, the above equation implies that χ_+ is a Killing spinor and that the length $\|\chi_+\| = \text{const}$.

A similar formula to (3.7.5) can be established for σ_- and τ_- spinors. However, it is not necessary to do an independent commutation. We have seen that if σ_+ and τ_+ solve the KSEs, then $\sigma_- = A\Gamma_{-z}\sigma_+$ and $\tau_- = A\Gamma_{-z}\tau_+$ also solve the KSEs. Similarly if χ_+ is a zero mode of $\mathcal{D}^{(+)}$, then $\chi_- = A\Gamma_{-z}\chi_+$ is a zero mode of $\mathcal{D}^{(-)}$, where

$$\mathcal{D}^{(-)} = \Gamma^i \nabla_i + \Sigma^{(-)} , \quad (3.7.6)$$

and

$$\Sigma^{(-)} = -\frac{3qc}{2\ell} A^{-1} \Gamma_z + \frac{-1+6q}{4} \not\partial \log A^2 - \frac{i}{2} \not\partial - (2i-3iq) \not\Gamma_{zx} + \frac{1-q}{16} \not\Gamma C * . \quad (3.7.7)$$

To summarize, we have shown that

$$\begin{aligned} \nabla_i^{(\pm)} \sigma_{\pm} = 0 , \quad \mathcal{B}^{(\pm)} \sigma_{\pm} = 0 , \quad \mathcal{A}^{(\pm)} \sigma_{\pm} = 0 &\iff \mathcal{D}^{(\pm)} \sigma_{\pm} = 0 ; \quad c = 1 , \\ \nabla_i^{(+)} \tau_{\pm} = 0 , \quad \mathcal{C}^{(\pm)} \tau_{\pm} = 0 , \quad \mathcal{A}^{(\pm)} \tau_{\pm} = 0 &\iff \mathcal{D}^{(\pm)} \tau_{\pm} = 0 ; \quad c = -1 , \end{aligned} \quad (3.7.8)$$

and that

$$\begin{aligned} \|\sigma_+\| &= \text{const} , \quad \|\tau_+\| = \text{const} , \\ A^{-2} \|\sigma_-\|^2 &= \text{const} , \quad A^{-2} \|\tau_-\|^2 = \text{const} . \end{aligned} \quad (3.7.9)$$

This concludes the proof of the 1-1 correspondence between Killing spinors and zero modes of Dirac-like operators on M^6 .

3.7.2 Counting supersymmetries again

We are ready now to establish the formula for the number of preserved supersymmetries for AdS_4 backgrounds. So provided that the data satisfy the requirements of Hopf maximum principle, we have that

$$N = 4 \dim \text{Ker}(\nabla^{(-)}, \mathcal{A}^{(-)}, \mathcal{B}^{(-)}) = 4 \dim \text{Ker} \mathcal{D}_{c=1}^{(-)} , \quad (3.7.10)$$

which applies to σ_- spinors which confirms $N = 4k$. A similar formula is valid for the three other choice of spinors.

3.8 AdS_5 : Local analysis

3.8.1 Fields, Bianchi identities and field equations

The fields on the horizon section \mathcal{S} are

$$\begin{aligned} ds^2(\mathcal{S}) &= A^2(dz^2 + e^{\frac{2z}{\ell}}(dx^2 + dy^2)) + ds^2(M^5), \quad G = H, \quad P = \xi, \\ \tilde{F}^3 &= e^{\frac{2z}{\ell}} A^3 dz \wedge dx \wedge dy Y, \quad \tilde{F}^5 = -d\text{vol}(M^5)Y \end{aligned} \quad (3.8.1)$$

and $h = -\frac{2}{\ell}dz - 2d\log A$ and $\Delta = X = L = 0$.

Substituting the above fields into the Bianchi identities (3.1.15) and (3.1.16), we find

$$\begin{aligned} d(A^5 Y) &= 0, \quad dH = iQ \wedge H - \xi \wedge \bar{H} \\ d\xi &= 2iQ \wedge \xi, \quad dQ = -i\xi \wedge \bar{\xi}. \end{aligned} \quad (3.8.2)$$

Clearly, Y is proportional to A^{-5} . Similarly, the field equations (3.1.17)-(3.1.21) give

$$\begin{aligned} \nabla^i H_{ijk} &= -5\partial^i \log A H_{ijk} + iQ^i H_{ijk} + \xi^i \bar{H}_{ijk} \\ \nabla^i \xi_i &= -5\partial^i \log A \xi_i + 2iQ^i \xi_i - \frac{1}{24} H^2, \\ A^{-1} \nabla^2 A &= 4Y^2 + \frac{1}{48} \|H\|^2 - \frac{4}{\ell^2} A^{-2} - 4(d\log A)^2, \\ R_{ij}^{(5)} &= 5A^{-1} \nabla_i \nabla_j A + 4Y^2 \delta_{ij} \\ &\quad + \frac{1}{4} H_{(i}{}^{k\ell} \bar{H}_{j)k\ell} - \frac{1}{48} \|H\|^2 \delta_{ij} + 2\xi_{(i} \bar{\xi}_{j)}. \end{aligned} \quad (3.8.3)$$

This concludes the analysis of Bianchi and field equations.

The warped factor is nowhere vanishing

As in the previous AdS backgrounds, one can show that the warped factor A is no-where vanishing. The argument is based on the third field equation in (3.8.3).

3.8.2 Solution of KSEs

Substituting the fields of the previous section into the KSEs of spatial horizon section (3.1.27) and after a computation similar to that described for AdS_4 backgrounds, we find that the Killing spinors can be expressed as

$$\eta_+ = \sigma_+ - \frac{1}{\ell}(x\Gamma_x + y\Gamma_y)\Gamma_z \tau_+ + e^{-\frac{z}{\ell}} \tau_+, \quad \eta_- = \sigma_- + e^{\frac{z}{\ell}} \left(-\frac{1}{\ell}(x\Gamma_x + y\Gamma_y)\Gamma_z \sigma_- + \tau_- \right), \quad (3.8.4)$$

where σ_{\pm} and τ_{\pm} dependent only on the coordinates of M^5 . The remaining independent KSEs are

$$\begin{aligned} \nabla_i^{(\pm)} \sigma_{\pm} &= 0, \quad \nabla_i^{(\pm)} \tau_{\pm} = 0, \quad \mathcal{A}^{(\pm)} \sigma_{\pm} = 0, \quad \mathcal{A}^{(\pm)} \tau_{\pm} = 0, \\ \mathcal{B}^{\pm} \sigma_{\pm} &= 0, \quad \mathcal{C}^{\pm} \tau_{\pm} = 0, \end{aligned} \quad (3.8.5)$$

where

$$\begin{aligned}\nabla_i^{(\pm)} &= \nabla_i + \Psi_i^{(\pm)}, \quad \mathcal{A}^{(\pm)} = \frac{1}{24}\not{H} + \not{C}^* \\ \mathcal{B}^{(\pm)} &= \Xi_{\pm}, \quad \mathcal{C}^{(\pm)} = \Xi_{\pm} \pm \frac{1}{\ell},\end{aligned}\tag{3.8.6}$$

and

$$\begin{aligned}\Psi_i^{(\pm)} &= \pm \frac{1}{2} \partial_i \log A - \frac{i}{2} Q_i \pm \frac{i}{2} \Gamma_i Y \Gamma_{xyz} + \left(-\frac{1}{96} (\Gamma \not{H})_i + \frac{3}{32} \not{H}_i \right) C^* \\ \Xi_{\pm} &= \mp \frac{1}{2\ell} - \frac{1}{2} \Gamma_z \not{A} \pm \frac{i}{2} A Y \Gamma_{xy} + \frac{1}{96} A \Gamma_z \not{H} C^*.\end{aligned}\tag{3.8.7}$$

This concludes the solution of the KSEs along the AdS_5 directions and the identification of remaining independent KSEs.

3.8.3 Counting supersymmetries

To count the number of supersymmetries preserved by AdS_5 backgrounds, observe that if σ_{\pm} are Killing spinors, then

$$\tau_{\pm} = \Gamma_z \Gamma_x \sigma_{\pm}, \quad \tau_{\pm} = \Gamma_z \Gamma_y \sigma_{\pm}.\tag{3.8.8}$$

are also Killing spinors, and vice versa. As a result if σ_{\pm} are Killing spinors, then $\sigma'_{\pm} = \Gamma_{xy} \sigma_{\pm}$ are also Killing spinors and similarly for τ_{\pm} . As a result $\dim \text{Ker}(\nabla^{(\pm)}, \mathcal{A}^{(\pm)}, \mathcal{B}^{(\pm)})$ and $\dim \text{Ker}(\nabla^{(\pm)}, \mathcal{A}^{(\pm)}, \mathcal{C}^{(\pm)})$ are even numbers.

Furthermore, as in the previous cases, if either σ_- or τ_- is a solution, so is

$$\sigma_+ = A^{-1} \Gamma_+ \Gamma_z \sigma_-, \quad \tau_+ = A^{-1} \Gamma_+ \Gamma_z \tau_-, \tag{3.8.9}$$

and similarly, if either σ_+ or τ_+ is a solution, so is

$$\sigma_- = A \Gamma_- \Gamma_z \sigma_+, \quad \tau_- = A \Gamma_- \Gamma_z \tau_+.\tag{3.8.10}$$

From the above relations one concludes that the $AdS_5 \times_w M^5$ backgrounds preserve

$$N = 4 \dim \text{Ker}(\nabla^{(\pm)}, \mathcal{A}^{(\pm)}, \mathcal{B}^{(\pm)}) = 4 \dim \text{Ker}(\nabla^{(\pm)}, \mathcal{A}^{(\pm)}, \mathcal{C}^{(\pm)}) = 8k, \tag{3.8.11}$$

for either $+$ or $-$ choice of sign and $k \in \mathbb{N}_{>0}$. This confirms $N = 8k$ for the AdS_5 backgrounds. Of course $N \leq 32$, and for $N = 32$ the solutions are locally isometric to $AdS_5 \times S^5$.

3.9 AdS_5 : Global analysis

3.9.1 A Lichnerowicz theorem for τ_{\pm} and σ_{\pm}

To extend the formula for preserved supersymmetries to AdS_5 backgrounds, we shall again prove a Lichnerowicz type of theorem which relates the Killing spinors to the zero modes of Dirac-like

of operators on M^5 coupled to fluxes. The proof is similar to that we have presented in previous case and so we shall be brief. It suffices to prove the relation for σ_+ and τ_+ spinors as the proof for the other pair σ_- and τ_- follows because of the relations (3.8.9) and (3.8.10) and the fact that these isomorphisms commute with the relevant operators.

We shall present the proof of the the Lichnerowicz type of theorem for σ_+ and τ_+ spinors. The proof for the other pair σ_- and τ_- follows as a consequence. It is also convenient to do the computations simultaneously for both σ_+ and τ_+ spinors which from now on we shall call collectively χ_+ .

To begin the proof for the pair σ_+ and τ_+ , which from now on we shall call collectively χ_+ , let us define

$$\mathbb{D}_i^{(+)} = \nabla_i^{(+)} + q\Gamma_{zi}A^{-1}\mathbb{B}^{(+)} \quad (3.9.1)$$

where

$$\mathbb{B}^{(+)} = -\frac{c}{2\ell} - \frac{1}{2}\Gamma_z\partial A - \frac{i}{2}AY\Gamma_{yx} + \frac{1}{96}A\Gamma_z\#C* \quad (3.9.2)$$

and $c = 1$ when acting on σ_+ and $c = -1$ when acting on τ_+ , ie either $\mathbb{B}^{(+)} = \mathcal{B}^{(+)}$ or $\mathbb{B}^{(+)} = \mathcal{C}^{(+)}$, respectively. It is clear from this that if χ_+ is a Killing spinor, then it is parallel with respect to \mathbb{D} .

The Dirac-like operator on M^5 is

$$\mathcal{D}^{(+)} \equiv \Gamma^i\mathbb{D}_i^{(+)} = \Gamma^i\nabla_i + \Sigma^{(+)} . \quad (3.9.3)$$

where

$$\Sigma^{(+)} = \frac{5qc}{2\ell}A^{-1}\Gamma_z + \frac{1+5q}{4}\partial\log A^2 - \frac{i}{2}\mathcal{Q} + \frac{5i-5iq}{2}Y\Gamma_{zxy} + \frac{7-5q}{96}\#C* . \quad (3.9.4)$$

Next suppose that χ_+ is a zero mode of $\mathcal{D}^{(+)}$, ie $\mathcal{D}^{(+)}\chi_+ = 0$. Then after some Clifford algebra computation, which has been presented in appendix B.14, $q = 3/5$, and the use of field equations, one can establish the identity

$$\begin{aligned} \nabla^2\|\chi_+\|^2 + 5A^{-1}\partial^i A\partial_i\|\chi_+\|^2 &= 2\left\|\mathbb{D}^{(+)}\chi_+\right\|^2 \\ &+ \frac{48}{5}A^{-2}\left\|\mathbb{B}^{(+)}\chi_+\right\|^2 + \left\|\mathcal{A}^{(+)}\chi_+\right\|^2 . \end{aligned} \quad (3.9.5)$$

Assuming that the Hopf maximum principle applies, eg for M^5 compact, the solution of the above equation reveals that χ_+ is a Killing spinor and that $\|\chi_+\| = \text{const}$.

A similar formula to (3.9.5) can be established for σ_- and τ_- spinors. In particular, we define

$$\mathcal{D}^{(-)} = \Gamma^i\nabla_i + \Sigma^{(-)} , \quad (3.9.6)$$

and

$$\Sigma^{(-)} = -\frac{5qc}{2\ell}A^{-1}\Gamma_z + \frac{-1+5q}{4}\partial\log A^2 - \frac{i}{2}\mathcal{Q} - \frac{5i-5iq}{2}Y\Gamma_{zxy} + \frac{7-5q}{96}\#C* . \quad (3.9.7)$$

where $c = 1$ for the σ_- spinors while $c = -1$ for τ_- spinors. Because of the relations (3.8.9) and (3.8.10) between the σ_- , τ_- and σ_+ , τ_+ spinors and the commutation of these relations with the KSEs and the associated Dirac-like operators, it is not necessary to prove the maximum principle independently for σ_- , τ_- . To summarize, we have shown that

$$\begin{aligned} \nabla_i^{(\pm)} \sigma_{\pm} = 0, \quad \mathcal{B}^{(\pm)} \sigma_{\pm} = 0, \quad \mathcal{A}^{(\pm)} \sigma_{\pm} = 0 &\iff \mathcal{D}^{(\pm)} \sigma_{\pm} = 0; \quad c = 1, \\ \nabla_i^{(\pm)} \tau_{\pm} = 0, \quad \mathcal{C}^{(\pm)} \tau_{\pm} = 0, \quad \mathcal{A}^{(\pm)} \tau_{\pm} = 0 &\iff \mathcal{D}^{(\pm)} \tau_{\pm} = 0; \quad c = -1, \end{aligned} \quad (3.9.8)$$

and that

$$\begin{aligned} \|\sigma_+\| &= \text{const}, \quad \|\tau_+\| = \text{const}, \\ A^{-2} \|\sigma_-\|^2 &= \text{const}, \quad A^{-2} \|\tau_-\|^2 = \text{const}. \end{aligned} \quad (3.9.9)$$

3.9.2 Counting supersymmetries again

To determine the index of the Dirac-like operator for AdS_5 backgrounds, observe that the dimension of the Kernel of $\mathcal{D}^{(\pm)}$ operators is even. This is because if σ_{\pm} or τ_{\pm} are in the kernel, then $\Gamma_{xy}\sigma_{\pm}$ or $\Gamma_{xy}\tau_{\pm}$ are also in the kernel. Since $\Gamma_{xy}\sigma_{\pm}$ or $\Gamma_{xy}\tau_{\pm}$ are linearly independent of σ_{\pm} and τ_{\pm} , the dimension of the Kernel of $\mathcal{D}^{(\pm)}$ is an even number.

Next provided that the data satisfy the requirements of Hopf maximum principle, we have that

$$N = 4 \dim \text{Ker}(\nabla^{(-)}, \mathcal{A}^{(-)}, \mathcal{B}^{(-)}) = 4 \dim \text{Ker} \mathcal{D}_{c=1}^{(-)}, \quad (3.9.10)$$

which applies to σ_- spinors. A similar formula is valid for the three other choices of spinors.

3.10 AdS_6 : Local analysis

3.10.1 Fields, Bianchi identities and field equations

For AdS_p , $p \geq 6$, the only not vanishing fluxes are those of the magnetic components of the various field strengths. Since F is self-dual, $F = 0$ for all such backgrounds. The fields on the horizon section \mathcal{S} for AdS_6 backgrounds are

$$ds^2(\mathcal{S}) = A^2(dz^2 + e^{\frac{2z}{\ell}} \sum_{a=1}^3 (dx^a)^2) + ds^2(M^4), \quad G = H, \quad P = \xi, \quad (3.10.1)$$

and $h = -\frac{2}{\ell}dz - 2d \log A$ and $\Delta = X = L = 0$, where $x^1 = x, x^2 = y$ as for AdS_5 and $x^3 = w$.

Substituting the above fields into the Bianchi identities (3.1.15) and (3.1.16), we find

$$dH = iQ \wedge H - \xi \wedge \bar{H}, \quad d\xi = 2iQ \wedge \xi, \quad dQ = -i\xi \wedge \bar{\xi}. \quad (3.10.2)$$

Similarly, the field equations (3.1.17)-(3.1.21) give

$$\nabla^i H_{ijk} = -6\partial^i \log A H_{ijk} + iQ^i H_{ijk} + \xi^i \bar{H}_{ijk}$$

$$\begin{aligned}
\nabla^i \xi_i &= -6\partial^i \log A \xi_i + 2iQ^i \xi_i - \frac{1}{24}H^2. \\
A^{-1}\nabla^2 A &= \frac{1}{48} \|H\|^2 - \frac{5}{\ell^2}A^{-2} - 5(d\log A)^2, \\
R_{ij}^{(4)} &= 6A^{-1}\nabla_i \nabla_j A \\
&+ \frac{1}{4}H_{(i}{}^{k\ell}\overline{H}_{j)k\ell} - \frac{1}{48}\|H\|^2 \delta_{ij} + 2\xi_{(i}\bar{\xi}_{j)}.
\end{aligned} \tag{3.10.3}$$

This concludes the analysis of Bianchi and field equations.

The warped factor is nowhere vanishing

As in the previous AdS backgrounds, one can show that the warped factor A is no-where vanishing. The argument is based on the third field equation in (3.10.3).

3.10.2 Solution of KSEs

The solution of the spatial horizon section \mathcal{S} KSEs (3.1.27) reveals that

$$\eta_+ = \sigma_+ - \frac{1}{\ell}(\sum_a x^a \Gamma_a) \Gamma_z \tau_+ + e^{-\frac{z}{\ell}} \tau_+, \quad \eta_- = \sigma_- + e^{\frac{z}{\ell}} \left(-\frac{1}{\ell}(\sum_a x^a \Gamma_a) \Gamma_z \sigma_- + \tau_- \right), \tag{3.10.4}$$

where σ_\pm and τ_\pm dependent only on the coordinates of M^4 . After taking into account all the integrability conditions, the remaining independent KSEs are

$$\begin{aligned}
\nabla_i^{(\pm)} \sigma_\pm &= 0, \quad \nabla_i^{(\pm)} \tau_\pm = 0, \quad \mathcal{A}^{(\pm)} \sigma_\pm = 0, \quad \mathcal{A}^{(\pm)} \tau_\pm = 0, \\
\mathcal{B}^\pm \sigma_\pm &= 0, \quad \mathcal{C}^\pm \tau_\pm = 0,
\end{aligned} \tag{3.10.5}$$

where

$$\begin{aligned}
\nabla_i^{(\pm)} &= \nabla_i + \Psi_i^{(\pm)}, \quad \mathcal{A}^{(\pm)} = \frac{1}{24}\not{H} + \not{\xi} C^*, \\
\mathcal{B}^{(\pm)} &= \Xi_\pm, \quad \mathcal{C}^{(\pm)} = \Xi_\pm \pm \frac{1}{\ell},
\end{aligned} \tag{3.10.6}$$

and

$$\begin{aligned}
\Psi_i^{(\pm)} &= \pm \frac{1}{2} \partial_i \log A - \frac{i}{2} Q_i + \left(-\frac{1}{96} (\Gamma \not{H})_i + \frac{3}{32} \not{H}_i \right) C^*, \\
\Xi_\pm &= \mp \frac{1}{2\ell} - \frac{1}{2} \Gamma_z \partial_i A \Gamma^i + \frac{1}{96} A \Gamma_z \not{H} C^*.
\end{aligned} \tag{3.10.7}$$

This concludes the solution of the KSEs along the AdS_6 directions.

3.10.3 Counting supersymmetries

A direct inspection of the KSEs reveals that if σ_\pm are Killing spinors, then

$$\tau_\pm = \Gamma_z \Gamma_a \sigma_\pm, \tag{3.10.8}$$

are also Killing spinors, and vice versa. As a result if σ_{\pm} are Killing spinors, then $\sigma'_{\pm} = \Gamma_{ab}\sigma_{\pm}$ are also Killing spinors and similarly for τ_{\pm} . Therefore $\dim \text{Ker}(\nabla^{(\pm)}, \mathcal{A}^{(\pm)}, \mathcal{B}^{(\pm)})$ and $\dim \text{Ker}(\nabla^{(\pm)}, \mathcal{A}^{(\pm)}, \mathcal{C}^{(\pm)})$ are multiples of four.

Furthermore, as in the previous cases, if either σ_{-} or τ_{-} is a solution, so is

$$\sigma_{+} = A^{-1}\Gamma_{+}\Gamma_z\sigma_{-} , \quad \tau_{+} = A^{-1}\Gamma_{+}\Gamma_z\tau_{-} , \quad (3.10.9)$$

and similarly, if either σ_{+} or τ_{+} is a solution, so is

$$\sigma_{-} = A\Gamma_{-}\Gamma_z\sigma_{+} , \quad \tau_{-} = A\Gamma_{-}\Gamma_z\tau_{+} . \quad (3.10.10)$$

From the above relations one concludes that the $AdS_5 \times_w M^5$ backgrounds preserve

$$N = 4 \dim \text{Ker}(\nabla^{(\pm)}, \mathcal{A}^{(\pm)}, \mathcal{B}^{(\pm)}) = 4 \dim \text{Ker}(\nabla^{(\pm)}, \mathcal{A}^{(\pm)}, \mathcal{C}^{(\pm)}) = 16k , \quad (3.10.11)$$

for either $+$ or $-$ choice of sign and $k \in \mathbb{N}_{>0}$. This confirms $N = 16k$ for the AdS_6 backgrounds. It turns out that there can be AdS_6 backgrounds for only $N = 16$ as there are no such backgrounds preserving $N = 32$ supersymmetries [37].

3.11 AdS_6 : Global analysis

3.11.1 A Lichnerowicz theorem for τ_{\pm} and σ_{\pm}

As in previous cases, let us prove a Lichnerowicz type of theorem for σ_{+} and τ_{+} spinors. For this denote σ_{+} and τ_{+} collectively by χ_{+} and define

$$\mathbb{D}_i^{(+)} = \nabla_i^{(+)} + q\Gamma_{zi}A^{-1}\mathbb{B}^{(+)} \quad (3.11.1)$$

where

$$\mathbb{B}^{(+)} = -\frac{c}{2\ell} - \frac{1}{2}\Gamma_z\mathcal{Q}A\Gamma^i + \frac{1}{96}A\Gamma_z\mathcal{H}C* \quad (3.11.2)$$

and $c = 1$ when acting on σ_{+} and $c = -1$ when acting on τ_{+} , ie either $\mathbb{B}^{(+)} = \mathcal{B}^{(+)}$ or $\mathbb{B}^{(+)} = \mathcal{C}^{(+)}$, respectively. It is clear from this that if χ_{+} is a Killing spinor, then it is parallel with respect to \mathbb{D} .

The Dirac-like operator on M^4 is

$$\mathcal{D}^{(+)} \equiv \Gamma^i\mathbb{D}_i^{(+)} = \Gamma^i\nabla_i + \Sigma^{(+)} . \quad (3.11.3)$$

where

$$\Sigma^{(+)} = \frac{2qc}{\ell}A^{-1}\Gamma_z + \frac{1+4q}{4}\mathcal{Q}\log A^2 - \frac{i}{2}\mathcal{Q} + \frac{8-4q}{96}\mathcal{H}C* . \quad (3.11.4)$$

Next suppose that χ_{+} is a zero mode of $\mathcal{D}^{(+)}$, ie $\mathcal{D}^{(+)}\chi_{+} = 0$. Then after some Clifford algebra computation, which has been presented in appendix B.15, $q = 1$, and the use of field equations, one can establish the identity

$$\nabla^2\|\chi_{+}\|^2 + 6A^{-1}\partial^i A\partial_i\|\chi_{+}\|^2 = 2\|\mathbb{D}^{(+)}\chi_{+}\|^2$$

$$+16A^{-2}\left\|\mathbb{B}^{(+)}\chi_+\right\|^2 + \left\|\mathcal{A}^{(+)}\chi_+\right\|^2 . \quad (3.11.5)$$

Assuming that the Hopf maximum principle applies, eg for M^4 compact, the solution of the above equation reveals that χ_+ is a Killing spinor and that $\|\chi_+\| = \text{const}$.

A similar formula to (3.9.5) can be established for σ_- and τ_- spinors. In particular, we define

$$\mathcal{D}^{(-)} = \Gamma^i \nabla_i + \Sigma^{(-)} , \quad (3.11.6)$$

and

$$\Sigma^{(-)} = -\frac{2qc}{\ell}A^{-1}\Gamma_z + \frac{-1+4q}{4}\not\partial \log A^2 - \frac{i}{2}\not{Q} + \frac{8-4q}{96}\not{H}C * . \quad (3.11.7)$$

where $c = 1$ for the σ_- spinors while $c = -1$ for τ_- spinors. Because of the relations (3.8.9) and (3.8.10) between the σ_- , τ_- and σ_+ , τ_+ spinors and the commutation of these relations with the KSEs and the associated Dirac-like operators, it is not necessary to prove the maximum principle independently for σ_- , τ_- . To summarize, we have shown that

$$\begin{aligned} \nabla_i^{(\pm)}\sigma_{\pm} = 0 , \quad \mathcal{B}^{(\pm)}\sigma_{\pm} = 0 , \quad \mathcal{A}^{(\pm)}\sigma_{\pm} = 0 &\iff \mathcal{D}^{(\pm)}\sigma_{\pm} = 0 ; \quad c = 1 , \\ \nabla_i^{(+)}\tau_{\pm} = 0 , \quad \mathcal{C}^{(\pm)}\tau_{\pm} = 0 , \quad \mathcal{A}^{(\pm)}\tau_{\pm} = 0 &\iff \mathcal{D}^{(\pm)}\tau_{\pm} = 0 ; \quad c = -1 , \end{aligned} \quad (3.11.8)$$

and that

$$\begin{aligned} \|\sigma_+\| &= \text{const} , \quad \|\tau_+\| = \text{const} , \\ A^{-2} \|\sigma_-\|^2 &= \text{const} , \quad A^{-2} \|\tau_-\|^2 = \text{const} . \end{aligned} \quad (3.11.9)$$

3.11.2 Counting supersymmetries again

To determine the index of the Dirac-like operator for AdS_5 backgrounds, observe that the dimension of the Kernel of $\mathcal{D}^{(\pm)}$ operators is multiple of 4. This is because if σ_{\pm} or τ_{\pm} are in the kernel, then $\Gamma_{ab}\sigma_{\pm}$ or $\Gamma_{ab}\tau_{\pm}$ are also in the kernel. Since $\Gamma_{ab}\sigma_{\pm}$ or $\Gamma_{ab}\tau_{\pm}$ are linearly independent of σ_{\pm} and τ_{\pm} , the dimension of the Kernel of $\mathcal{D}^{(\pm)}$ is $4k$.

Next provided that the data satisfy the requirements of Hopf maximum principle, we have that

$$N = 4 \dim \text{Ker}(\nabla^{(-)}, \mathcal{A}^{(-)}, \mathcal{B}^{(-)}) = 4 \dim \text{Ker} \mathcal{D}_{c=1}^{(-)} = 16k , \quad (3.11.10)$$

which applies to σ_- spinors. A similar formula is valid for the three other choices of spinors.

3.12 AdS_n , for $n \geq 7$

For all AdS_n , $n \geq 7$, if the background preserves at least one supersymmetry, then the three-form, H , is zero. For AdS_n , $n \geq 8$, this is automatically true. For AdS_7 , we can show this by manipulating the algebraic Killing spinor equation,

$$\left(\not{H}C * + \frac{1}{24}\not{H}\right)\sigma_+ = 0. \quad (3.12.1)$$

$AdS_n \times_w M^{10-n}$	N
$n = 2$	$2k, k < 14$
$n = 3$	$2k, k < 14$
$n = 4$	$4k, k < 7$
$n = 5$	$8k, k \leq 4$
$n = 6$	16
$n \geq 7$	—

Table 3.1: The number of supersymmetries N of $AdS_n \times_w M^{10-n}$ backgrounds are given. For $AdS_2 \times_w M^8$, one can show that these backgrounds preserve even number of supersymmetries provided that they are smooth and M^8 is compact without boundary. For the rest, the counting of supersymmetries does not rely on the compactness of M^{10-n} . The bounds in k arise from the non-existence of supersymmetric solutions with near maximal and maximal supersymmetry. For the remaining fractions, it is not known whether there always exist backgrounds preserving the prescribed number of supersymmetries. Supersymmetric AdS_n , $n \geq 7$, backgrounds do not exist.

We start by multiplying this by \bar{H} to convert it to an eigenvalue equation,

$$\not{g} \bar{H} C * \sigma_+ = -\frac{1}{4} \|H\|^2 \sigma_+, \quad (3.12.2)$$

and then we square the operator on the left hand side to eliminate $C*$,

$$\xi_i \bar{\xi}_j \Gamma^{ij} \sigma_+ = -\left(\|\xi\|^2 + \frac{1}{96} \|H\|^2\right) \sigma_+. \quad (3.12.3)$$

Finally, squaring this operator as well, we end up with a scalar equation

$$\|\xi\|^4 - \|\xi^2\|^2 = \left(\|\xi\|^2 + \frac{1}{96} \|H\|^2\right)^2, \quad (3.12.4)$$

from which we conclude that $\xi^2 = \xi_i \xi^i$ and H are both zero.

Having shown that $H = 0$, the integrability condition, (3.1.31), reduces to

$$\left(\frac{1}{4\ell^2} + \frac{1}{4}(dA)^2\right) \sigma_+ = 0, \quad (3.12.5)$$

which has no solution. Therefore, there no supersymmetric AdS_n backgrounds, for $n \geq 7$.

3.13 Flat IIB backgrounds

Warped flat backgrounds $\mathbb{R}^{n-1,1} \times_w M^{10-n}$ are also included in our analysis. These arise in the “flat limit”, ie the limit that the AdS_n radius ℓ is taken to infinity. This limit is smooth in

all our computations. However, some of our results on AdS_n backgrounds do not extend to the flat backgrounds. The investigation of the KSEs is also somewhat different from that of AdS backgrounds.

To emphasize some of the differences between AdS_n and $\mathbb{R}^{n-1,1}$ backgrounds, it is known for sometime that there are no smooth warped flux compactification in the supergravity [40]. To alter this either additional sources have to be added to the supergravity equations, like brane charges, and /or consider higher order curvature corrections which arise for example from anomaly cancelation mechanisms or α' corrections in strings or M-theory. In either case, the new backgrounds can be constructed as corrections to supergravity solutions. Because there are different sources that can be added and we do not have control over all higher curvature corrections, we shall mostly focus here on the supergravity limit and explore the similarities and differences between the AdS_n and $\mathbb{R}^{n-1,1}$ backgrounds.

3.13.1 Warped factor is not nowhere vanishing

We have seen that the warped factor in all AdS_n is no-where vanishing. This does not extend to $\mathbb{R}^{n-1,1}$ backgrounds because the finiteness of AdS_n radius has been essential in the proof of the statement. In fact A must vanish somewhere for non-trivial $\mathbb{R}^{n-1,1}$ backgrounds with fluxes. This follows from the results of [40] on the non-existence of smooth warped flux compactifications in the context of supergravity. To see this, let us focus on the $\mathbb{R}^{1,1}$ case, as the argument is similar in all the other cases. If A is no-where vanishing and M^8 is compact, an application of the maximum principle on the field equation for A (3.2.8) reveals that A is constant and the fluxes F and G vanish. Furthermore using the formula

$$\nabla^2 \|\xi\|^2 = 2(\nabla_{(i}\xi_{j)} - 2i\Lambda_{(i}\xi_{j)})(\nabla^{(i}\xi^{j)} - 2i\Lambda^{(i}\xi^{j)}) + 6(\|\xi\|^2)^2 \quad (3.13.1)$$

established [11] and upon using again the maximum principle, one can show that $\xi = 0$. As a result all the form field strengths vanish which is a contradiction. From now on, we shall assume that A is non-vanishing on some dense subset of M^{10-n} and carry out the analysis that follows on that subset.

3.13.2 Counting supersymmetries

All the local computations we have done for AdS_n backgrounds extend to $\mathbb{R}^{n-1,1}$ backgrounds. However the statements which rely on the smoothness of the fields as well as the non-vanishing of the warped factor have to be re-examined. In particular, the solution of the KSEs can be carried out as it has been described for AdS_n . Also the various maximum principle formulae are valid away from points that $A = 0$, like eg (3.3.3), (3.3.5), (3.5.6) and others. However, the Hopf maximum principle cannot be applied any longer even if M^{10-n} is taken to be compact. As a result there is not a straightforward relation between Killing spinors and zero modes on Dirac-like operators on M^{10-n} . Because of this, for the counting of supersymmetries we shall rely on the local solution of the KSEs as presented for the AdS_n backgrounds.

$\mathbb{R}^{1,1}$ backgrounds

The counting of supersymmetries for AdS_2 backgrounds relies on the global properties of M^8 and the smoothness of the fields. As a result, the number of supersymmetries preserved by $\mathbb{R}^{1,1}$ backgrounds cannot be concluded. In particular, it is not apparent that such backgrounds always preserve even number of supersymmetries. Nevertheless, if η_- is a Killing spinor, so is $\Gamma_+ \Theta_- \eta_-$ on M^8 . Now if $\text{Ker} \Theta_- = \{0\}$, it is clear that there will be a doubling of supersymmetries. In such case, the number of Killing spinors for such backgrounds is $N \geq 2N_-$, where N_- is the number of η_- Killing spinors.

$\mathbb{R}^{2,1}$ backgrounds

Let us re-examine the solution of the KSEs. In the limit $\ell \rightarrow \infty$, the integrability conditions (3.1.31) become

$$\Theta_{\mp} \Theta_{\pm} \eta_{\pm} = 0 . \quad (3.13.2)$$

In the same limit, the solution of the KSEs (3.4.14) along the z-direction is

$$\eta_{\pm} = \sigma_{\pm} + z \Xi_{\pm} \tau_{\pm} , \quad \Xi_{\pm} (\sigma_{\pm} - \tau_{\pm}) = 0 , \quad (3.13.3)$$

where $\Xi_{\pm} = A \Gamma_z \Theta_{\pm}$. The integrability conditions are automatically satisfied because of (3.13.2). The remaining independent KSEs are

$$\begin{aligned} \nabla_i^{(\pm)} \sigma_{\pm} &= 0 , & \nabla_i^{(\pm)} \tau_{\pm} &= 0 , \\ \mathcal{A}^{(\pm)} \sigma_{\pm} &= 0 , & \mathcal{A}^{(\pm)} \tau_{\pm} &= 0 , \end{aligned} \quad (3.13.4)$$

where $\nabla^{(\pm)}$ and $\mathcal{A}^{(\pm)}$ are given in (3.4.21). As τ_{\pm} and σ_{\pm} satisfy the same differential equations are not linearly independent. As a result, it suffices to consider only the σ_{\pm} spinors and set $\tau_{\pm} = \sigma_{\pm}$. Therefore the number of supersymmetries preserved by $\mathbb{R}^{2,1}$ backgrounds is $N = \dim \text{Ker}(\nabla^{(+)}, \mathcal{A}^{(+)}) + \dim \text{Ker}(\nabla^{(-)}, \mathcal{A}^{(-)})$.

Next, it is straightforward to observe that if σ_- solution of (3.4.20) in the limit $\ell = \infty$, then

$$\sigma_+ = A^{-1} \Gamma_z \Gamma_+ \sigma_- , \quad (3.13.5)$$

is also a solution. Conversely, if σ_+ is a solution, then

$$\sigma_- = A \Gamma_z \Gamma_- \sigma_+ , \quad (3.13.6)$$

is also a solution. Therefore, $\dim \text{Ker}(\nabla^{(+)}, \mathcal{A}^{(+)}) = \dim \text{Ker}(\nabla^{(-)}, \mathcal{A}^{(-)})$, and so the $\mathbb{R}^{2,1}$ backgrounds preserve even number of supersymmetries.

Observe that in general the Killing spinors can depend non-trivially on the z coordinate. This is possible only if $\sigma_{\pm} \notin \text{Ker} \Xi_{\pm}$ even though it is required that $\sigma_{\pm} \in \text{Ker} \Xi_{\pm}^2$ because of (3.13.2).

$\mathbb{R}^{3,1}$ backgrounds

The counting of supersymmetries of $\mathbb{R}^{3,1}$ backgrounds is similar to $\mathbb{R}^{2,1}$ solutions. In particular integrating the KSEs along the z and x directions we find that

$$\eta_{\pm} = \sigma_{\pm} + A(z\Gamma_z + x\Gamma_x)\Theta_{\pm}\tau_{\pm}, \quad \Theta_{\pm}(\sigma_{\pm} - \tau_{\pm}) = 0, \quad (3.13.7)$$

with σ_{\pm} and τ_{\pm} both in the kernel of $(\nabla^{(\pm)}, \mathcal{A}^{(\pm)})$ given in (3.6.15) and $\Xi_{\pm} = A\Gamma_z\Theta_{\pm}$. Therefore as in the $\mathbb{R}^{2,1}$ case these spinors are not linearly independent and so suffices to consider σ_{\pm} and set $\tau_{\pm} = \sigma_{\pm}$. In addition if σ_+ is a solution, so is $\Gamma_{zx}\sigma_+$. This together with the fact that if σ_+ is a solution so is $\sigma_- = A\Gamma_z\Gamma_-\sigma_+$, and vice versa if σ_- is a solution so is $\sigma_+ = A^{-1}\Gamma_z\Gamma_+\sigma_-$, one concludes that $\mathbb{R}^{3,1}$ backgrounds preserve $N = 4k$ supersymmetries.

Note again that the Killing spinors are allowed to depend linearly on the coordinates of $\mathbb{R}^{3,1}$. This is the case only if $\tau_{\pm} \notin \text{Ker } \Xi_{\pm}$ even though it is required that $\tau_{\pm} \in \text{Ker } \Xi_{\pm}^2$ because of (3.13.2).

$\mathbb{R}^{n-1,1}$, $n > 4$, backgrounds

As in the previous cases, one can prove that

$$\eta_{\pm} = \sigma_{\pm} + A\left(\sum_{\mu} x^{\mu}\Gamma_{\mu}\right)\Theta_{\pm}\tau_{\pm}, \quad \Theta_{\pm}(\sigma_{\pm} - \tau_{\pm}) = 0, \quad (3.13.8)$$

in the limit $\ell = \infty$, and that the only linearly independent Killing spinors are σ_{\pm} , where x^{μ} are all the coordinates of $\mathbb{R}^{n-1,1}$ apart from the lighcone ones u, r . Moreover, it suffices to count the linearly independent σ_+ spinors as the σ_- spinors can be constructed as $\sigma_- = A\Gamma_z\Gamma_-\sigma_+$ from the σ_+ ones, and vice versa because of the relation $\sigma_+ = A^{-1}\Gamma_z\Gamma_+\sigma_-$.

Next given a σ_+ Killing spinor, one can see by direct inspection of the KSEs on M^{10-n} that $\Gamma_{ab}\sigma_+$, $a < b$, are also Killing spinors, where Γ_a are the gamma matrices in directions orthogonal to $+, -$. It turns out that for $n = 5$, these are all linearly independent and therefore these backgrounds preserve $N = 8k$ supersymmetries.

For $n = 6$, apart from $\Gamma_{ab}\sigma_+$, $a < b$, observe that also $\Gamma_{a_1a_2a_3a_4}\sigma_+$, $a_1 < a_2 < a_3 < a_4$ also solve the KSEs on M^4 . However, there is a unique Clifford algebra element $\Gamma_{a_1a_2a_3a_4}$, $a_1 < a_2 < a_3 < a_4$, in this case and has eigenvalues ± 1 , and commutes with all the KSEs. Now if σ_+ is in one of the two eigenspaces, only four of the 7 Killing spinors $\{\sigma_+, \Gamma_{ab}\sigma_+ | a < b\}$ are linearly independent. Therefore the $\mathbb{R}^{5,1}$ backgrounds preserve $N = 8k$ supersymmetries.

Suppose now that $n = 7$. Given a Killing spinor σ_+ , then $\Gamma_{ab}\sigma_+$ and $\Gamma_{a_1a_2a_3a_4}\sigma_+$, $a_1 < a_2 < a_3 < a_4$, are also Killing spinors. There are five $\Gamma_{a_1a_2a_3a_4}$, $a_1 < a_2 < a_3 < a_4$ Clifford algebra operations in this case. Choose one say $\Gamma_{[4]}$. As in the previous case σ_+ can be in one of the eigenspaces of $\Gamma_{[4]}$. In such a case, only 8 of the previous 16 Killing spinors are linearly independent. Therefore, the $\mathbb{R}^{6,1}$ backgrounds preserve $N = 16k$ supersymmetries. Of course as a consequence of [37] the non-trivial $\mathbb{R}^{6,1}$ backgrounds preserve strictly 16 supersymmetries.

$\mathbb{R}^{n-1,1} \times_w M^{10-n}$	N
$n = 2$	$N \leq 16$
$n = 3$	$2k, k \leq 8$
$n = 4$	$4k, k \leq 4$
$n = 5$	8, 16
$n = 6$	8, 16
$n = 7$	16
$n = 8$	16
$n = 8$	—

Table 3.2: The number of supersymmetries N of $\mathbb{R}^{1,1} \times_w M^{10-n}$ is not a priori an even number. The corresponding statement for AdS_2 backgrounds is proven using global considerations which are not available in this case. For the rest, the counting of supersymmetries follows from the properties of KSEs. All backgrounds with $N > 16$ supersymmetries are locally isometric to $\mathbb{R}^{9,1}$.

Furthermore adapting the analysis of section 13 in the limit of infinite AdS radius, one finds that A must be constant, $H = 0$ and $\xi_i \xi^i = 0$.

Next take $n = 8$. Given a Killing spinor σ_+ , then $\Gamma_{ab}\sigma_+$, $\Gamma_{a_1 a_2 a_3 a_4}\sigma_+$, $a_1 < a_2 < a_3 < a_4$, and $\Gamma_{a_1 \dots a_6}\sigma_+$, $a_1 < \dots < a_6$ are also Killing spinors. All fifteen $\Gamma_{a_1 a_2 a_3 a_4}$, $a_1 < a_2 < a_3 < a_4$, Clifford algebra operators commute with the KSEs and have eigenvalues ± 1 . Taking a commuting pair of such operators, say $\Gamma_{[4]}$ and $\Gamma'_{[4]}$, and choosing σ_+ to lie in a common eigenspace of both these operators, only eight of the 32 spinors mentioned above are linearly independent. As a result, $\mathbb{R}^{7,1}$ backgrounds preserve $N = 16k$ supersymmetries. In fact non-trivial $\mathbb{R}^{7,1}$ backgrounds, like the D7-brane, preserve strictly 16 supersymmetries. Again for this backgrounds A is constant and $\xi_i \xi^i = 0$. Furthermore, it can be easily seen from the results of section 13 and after taking the AdS radius to infinity that there are no non-trivial $\mathbb{R}^{8,1}$ supersymmetric backgrounds.

3.14 On the factorization of Killing spinors

In many of the investigations of $AdS_n \times M^{10-n}$ backgrounds in IIB and other theories, it is assumed that the Killing spinors of the spacetime factorize into a product

$$\epsilon = \psi \otimes \chi, \quad (3.14.1)$$

where ψ is a Killing spinor on the AdS spaces satisfying the equation

$$\nabla_\mu \psi + \lambda \gamma_\mu \psi = 0, \quad (3.14.2)$$

and where ∇ and γ_μ are the spin connection and gamma matrices on AdS_n , respectively. Since, we have solved the KSEs on the whole spacetime, we can now test this hypothesis. To do this observe that if the hypothesis is correct, then ϵ also solves the (3.14.2). So it suffices to substitute our Killing spinors into (3.14.2) to see whether they are automatically satisfy it. This computation is similar that that we have done for M-theory in [13]. It turns out that the Killing spinors ϵ solve (3.14.2) iff

$$\Gamma_z \epsilon = \pm \epsilon . \quad (3.14.3)$$

However our Killing spinors do *not* satisfy this equation. As a result the original hypothesis is not valid in general.

To illustrate that (3.14.3) is restrictive, we shall test it against the supersymmetry counting for the $AdS_5 \times S^5$ background. It is known that this background preserves all 32 supersymmetries. It can be easily seen that to solve the algebraic KSEs for this background in (3.8.5) for the τ_+ spinor, one has to impose

$$\Gamma_{xy} \tau_+ = \pm i \tau_+ . \quad (3.14.4)$$

After choosing one of the signs, it is clear that the dimension of the space of solutions is 8 counted over the reals. The gravitino KSE is then solved without any additional constraints on τ_+ . Next using the relation between τ_+ , τ_- , σ_+ and σ_- solutions to the KSEs, we conclude that the number of Killing spinors of this background is $4 \times 8 = 32$ as expected. However if one also imposes the condition (3.14.3) on τ_+ , one will arrive at the incorrect conclusion that $AdS_5 \times S^5$ preserves only 16 supersymmetries.

We have seen that the spinor factorization assumption in (3.14.1) leads to the incorrect counting of supersymmetries for AdS backgrounds. It is also likely that it puts additional restrictions on the geometry of the transverse spaces M^{10-n} . We shall investigate this in another publication.

To continue, let us examine the factorization of the Killing spinors as in (3.14.1) for flat backgrounds to see whether a similar issue arises as for the AdS. A direct inspection of the Killing spinors we have found in section 3.13.2 reveals that the Killing spinors do not solve the KSEs on $\mathbb{R}^{n-1,1}$ whenever they have an explicit dependence on the coordinates of $\mathbb{R}^{n-1,1}$. As we have already stressed, this dependence appears whenever σ_\pm are not in the kernel of Θ_\pm . However it is required as a consequence of the KSEs, field equations and Bianchi identities that $\Theta_\mp \Theta_\pm \sigma_\pm = 0$. Thus assuming that the Killing spinor factorize as in (3.14.1) with ψ to be a constant spinor on $\mathbb{R}^{n-1,1}$, we find that this imposes the additional condition $\Theta_\pm \sigma_\pm = 0$ on the Killing spinors. It is not apparent that this condition always holds for flat backgrounds. On the other hand we are not aware of examples that it does not and so the question will be investigated further elsewhere.

3.15 Summary

In this chapter, the Killing spinor equations of $\text{AdS}_n \times_w M^{10-n}$ and $\mathbb{R}^{1,n-1} \times_w M^{10-n}$ IIB backgrounds have been solved. As a result, it was possible to determine the supersymmetry fractions preserved by these spaces. AdS_n backgrounds preserve $N = 2^{\lfloor \frac{n}{2} \rfloor} k$ supersymmetries for $n \leq 4$ and $N = 2^{\lfloor \frac{n}{2} \rfloor + 1} k$ supersymmetries for $4 < n \leq 6$. It has also been proven that there are no supersymmetric IIB AdS_n backgrounds for $n \geq 7$. $\mathbb{R}^{1,n-1}$ backgrounds preserve $N = 2^{\lfloor \frac{n}{2} \rfloor} k$ supersymmetries for $2 < n \leq 4$ and $N = 2^{\lfloor \frac{n+1}{2} \rfloor} k$ supersymmetries for $4 < n \leq 8$.

Often, when supersymmetric AdS backgrounds are discussed, it is assumed that the Killing spinors factorize into an AdS Killing spinor and a transverse Killing spinor [27, 28, 30, 41]. However, when this assumption is applied as an ansatz in addition to these results, the allowed supersymmetry fractions are further restricted. This indicates that the Killing spinors do not factorize in general, but only in special cases.

Additionally, for each AdS_n background, a Lichnerowicz-type theorem has been proven. These theorems assume that the transverse space satisfies the requirements of the Hopf maximum principle, which I use to prove that the σ_+ - and τ_+ -type Killing spinors are of constant length. Simultaneously, they prove that the Killing spinors are exactly the zero modes of a Dirac-like operator on the transverse space.

Chapter 4

IIA Backgrounds

IIA AdS backgrounds have also been of significant interest, both on their own and in relation to dual IIB and M-theory backgrounds [42, 30, 43, 44, 45, 46, 32]. Although they are similar to IIB backgrounds in many ways, the analysis in this chapter has some important differences from chapter 3. In particular, it is known that there are no maximally supersymmetric IIA AdS backgrounds.

The local analysis that is covered in this chapter demonstrates supersymmetry enhancement for all AdS backgrounds. AdS_n backgrounds are found to always have $N = 2^{\lfloor \frac{n}{2} \rfloor} k$ supersymmetries for $2 \leq n \leq 4$ and $N = 2^{\lfloor \frac{n}{2} \rfloor + 1} k$ supersymmetries for $5 \leq n \leq 7$, where $k \in \mathbb{Z}$. Note that for AdS_3 backgrounds, this means that supersymmetric backgrounds preserve $2k$ supersymmetries. In the next chapter, the same result will be demonstrated for heterotic backgrounds.

Additionally, I prove a Lichnerowicz-type theorem for each AdS background discussed, using methods similar to those in chapter 3. It's worth noting that there are some important differences between the algebra of these proofs and the IIB proofs. In particular, it is now necessary to include the dilatino Killing spinor equation in the Dirac-like operator.

4.1 $\text{AdS}_2 \times_w M^8$

4.1.1 Fields, Bianchi identities and Field Equations

Fields

As has already been mentioned, all AdS backgrounds are included in the near horizon geometries. To describe the fields of $\text{AdS}_2 \times_w M^8$ it suffices to impose the isometries of the AdS_2 space on all the fields of the near horizon geometries of [8, 9]. In such a case, the fields¹ can be written as

$$ds^2 = 2e^+e^- + ds^2(M^8) ,$$

¹The choice of the fields of $\text{AdS}_2 \times_w M^8$ backgrounds here is different from that of near horizon geometries in [8]. In particular all R-R fields have been multiplied by e^Φ . For more details see [47] and [48].

$$\begin{aligned}
G &= \mathbf{e}^+ \wedge \mathbf{e}^- \wedge X + Y, \quad H = \mathbf{e}^+ \wedge \mathbf{e}^- \wedge W + Z, \\
F &= \mathbf{e}^+ \wedge \mathbf{e}^- N + P, \quad S = S, \quad \Phi = \Phi.
\end{aligned} \tag{4.1.1}$$

where X and P are 2-forms on M^8 , Y is a 4-form on M^8 , Z is a 3-form on M^8 , and N and the dilaton Φ are functions on M^8 . $S = e^\Phi m$, where m is the mass parameter of massive IIA supergravity. For the standard IIA supergravity $m = 0$ and so $S = 0$. Furthermore,

$$\begin{aligned}
\mathbf{e}^+ &= du, \quad \mathbf{e}^- = (dr + rh - \frac{1}{2}r^2 \Delta du), \\
h &= -2A^{-1}dA = \Delta^{-1}d\Delta, \quad \Delta = \ell^{-2}A^{-2},
\end{aligned} \tag{4.1.2}$$

where the dependence on the coordinates u, r is explicit, A is the warp factor which depends only on the coordinates of M^8 and ℓ is the radius of AdS_2 .

Bianchi identities and Field equations

The Bianchi identities of (massive) IIA supergravity reduce to differential identities on the components of the fields localized on M^8 . In particular a direct computation reveals that

$$\begin{aligned}
d(A^2 W) &= 0, \quad d(A^2 X) - A^2 d\Phi \wedge X - A^2 W \wedge P - A^2 N Z = 0, \\
dZ &= 0, \quad d(A^2 N) - A^2 N d\Phi - S A^2 W = 0, \\
dY - d\Phi \wedge Y &= Z \wedge P, \quad dP - d\Phi \wedge P = SZ.
\end{aligned} \tag{4.1.3}$$

Similarly, the field equations of the (massive) IIA supergravity decomposed as

$$\begin{aligned}
\nabla^j P_{ji} + (2\partial^j \log A - \partial^j \Phi) P_{ji} - W^j X_{ji} + \frac{1}{6} Z^{jk\ell} Y_{jk\ell i} &= 0, \\
e^{2\Phi} \nabla^i (e^{-2\Phi} W_i) - SN - \frac{1}{2} P^{ij} X_{ij} + \frac{1}{48} * Y_{i_1 \dots i_4} Y^{i_1 \dots i_4} &= 0, \\
e^{2\Phi} \nabla^k (e^{-2\Phi} Z_{kij}) - S P_{ij} + 2\partial^k \log A Z_{kij} + N X_{ij} - \frac{1}{2} P^{kl} Y_{kl ij} \\
- \frac{1}{2} X_{k\ell} * Y_{ij}{}^{k\ell} &= 0, \\
\nabla^j X_{ji} - \partial^j \Phi X_{ji} + \frac{1}{6} * Y_i{}^{k_1 k_2 k_3} Z_{k_1 k_2 k_3} &= 0, \\
\nabla^i Y_{ijk\ell} + (2\partial^i \log A - \partial^i \Phi) Y_{ijk\ell} - \frac{1}{2} X_{m_1 m_2} * Z_{jk\ell}{}^{m_1 m_2} \\
- * Y_{jk\ell}{}^n W_n &= 0, \\
\nabla^2 \Phi + 2A^{-1} \partial^i A \partial_i \Phi = 2\partial^i \Phi \partial_i \Phi + \frac{1}{2} W^2 - \frac{1}{12} Z^2 - \frac{3}{4} N^2 \\
+ \frac{3}{8} P^2 - \frac{1}{8} X^2 + \frac{1}{96} Y^2 + \frac{5}{4} S^2, &
\end{aligned} \tag{4.1.4}$$

and in particular the Einstein equation decomposes as

$$\begin{aligned}
\nabla^i \partial_i \log A + \Delta + 2(d \log A)^2 &= 2\partial^i \log A \partial_i \Phi + \frac{1}{2} W^2 \\
+ \frac{1}{4} N^2 + \frac{1}{8} X^2 + \frac{1}{8} P^2 - \frac{1}{96} Y^2 - \frac{1}{4} S^2, \\
R_{ij}^{(8)} = 2\nabla_i \partial_j \log A + 2\partial_i \log A \partial_j \log A - 2\nabla_i \partial_j \Phi - \frac{1}{2} W_i W_j + \frac{1}{4} Z_{ij}^2 + \frac{1}{2} P_{ij}^2
\end{aligned}$$

$$-\frac{1}{2}X_{ij}^2 + \frac{1}{12}Y_{ij}^2 + \delta_{ij}\left(\frac{1}{4}N^2 - \frac{1}{4}S^2 - \frac{1}{8}P^2 + \frac{1}{8}X^2 - \frac{1}{96}Y^2\right), \quad (4.1.5)$$

where ∇ is the Levi-Civita connection on M^8 and the Latin indices i, j, k, \dots are frame M^8 indices.

4.1.2 Local aspects: Solutions of KSEs

Solution of KSEs along AdS_2

The solution of the KSEs for $\text{AdS}_2 \times_w M^8$ backgrounds is a special case of that presented for IIA horizons in [8]. In particular, the solution of the KSEs along the AdS_2 directions can be written as

$$\begin{aligned} \epsilon &= \epsilon_+ + \epsilon_- , \\ \epsilon_+ &= \eta_+ + u\Gamma_+ \Theta_- \eta_- , \quad \epsilon_- = \eta_- + r\Gamma_- \Theta_+ (\eta_+ + u\Gamma_+ \Theta_- \eta_-) , \end{aligned} \quad (4.1.6)$$

where $\Gamma_{\pm}\epsilon_{\pm} = 0$,

$$\Theta_{\pm} = -\frac{1}{2}A^{-1}\not{\partial}A \mp \Gamma_{11}\not{W} - \frac{1}{16}\Gamma_{11}(\pm 2N + \not{P}) - \frac{1}{8 \cdot 4!}(\pm 12\not{X} + \not{Y}) - \frac{1}{8}S , \quad (4.1.7)$$

and η_{\pm} depend only on the coordinates of M^8 . This summarizes the solution of the KSEs along the AdS_2 directions.

Independent KSEs on M^8

Having solved the KSEs along the AdS_2 directions, it remains to identify the remaining independent KSEs. This is not straightforward. After substituting (4.1.6) back into the KSEs of (massive) IIA supergravity and expanding in the u and r coordinates, one finds a large number of conditions. These can be interpreted as integrability conditions along the AdS_2 and mixed AdS_2 and M^8 directions. However after an extensive analysis which involves the use of Bianchi identities and field equations, one finds that the remaining independent KSEs are

$$\nabla_i^{(\pm)}\eta_{\pm} = 0 , \quad \mathcal{A}^{(\pm)}\eta_{\pm} = 0 , \quad (4.1.8)$$

where

$$\nabla_i^{(\pm)} = \nabla_i + \Psi_i^{(\pm)} , \quad (4.1.9)$$

and

$$\begin{aligned} \Psi_i^{(\pm)} &= \pm \frac{1}{2}A^{-1}\partial_i A \mp \frac{1}{16}\not{X}\Gamma_i + \frac{1}{8 \cdot 4!}\not{Y}\Gamma_i + \frac{1}{8}S\Gamma_i \\ &\quad + \Gamma_{11}\left(\mp \frac{1}{4}\not{W}_i + \frac{1}{8}\not{Z}_i \pm \frac{1}{8}\not{N}\Gamma_i - \frac{1}{16}\not{P}\Gamma_i\right) , \end{aligned} \quad (4.1.10)$$

and

$$\mathcal{A}^{(\pm)} = \not{\partial}\Phi + \left(\mp \frac{1}{8}\not{X} + \frac{1}{4 \cdot 4!}\not{Y} + \frac{5}{4}S\right)$$

$$+ \Gamma_{11} \left(\pm \frac{1}{2} \mathcal{W} - \frac{1}{12} \mathcal{Z} \mp \frac{3}{4} N + \frac{3}{8} \mathcal{P} \right) . \quad (4.1.11)$$

Furthermore, one can show that if η_- is a Killing spinor, ie satisfies (4.1.8), then

$$\eta_+ = \Gamma_+ \Theta_- \eta_- , \quad (4.1.12)$$

is also a Killing spinor.

Counting supersymmetries

The investigation so far is not sufficient to prove that the number of supersymmetries preserved by $\text{AdS}_2 \times_w M^8$ backgrounds is even. To prove this, some additional restrictions on the backgrounds are necessary which will be described in the next section.

4.1.3 Global aspects: Lichnerowicz type theorems

The non-vanishing of warp factor A

To proceed, we shall show that if A and the fields are smooth, then A does not vanish on M^8 . The argument which proves this is similar to that used in [13] and [14] to demonstrate the analogous statements for D=11 and IIB AdS backgrounds, and where a more detailed analysis is presented. Here we present a brief description of the proof which relies on the field equation of A . Assuming that A does not vanish everywhere on M , we multiply that field equation of A with A^2 at a value for which $A^2 \neq 0$ to find

$$\begin{aligned} -A \nabla^i \partial_i A - \ell^{-2} - \partial^i A \partial_i A = & -2A \partial^i A \partial_i \Phi - \frac{1}{2} A^2 \mathcal{W}^2 - \frac{1}{4} A^2 N^2 \\ & - \frac{1}{8} A^2 X^2 - \frac{1}{8} A^2 P^2 - \frac{1}{96} A^2 Y^2 - \frac{1}{4} A^2 S^2 . \end{aligned} \quad (4.1.13)$$

Then taking a sequence that converges to a point in M^8 that A vanishes, we find that if such a point exists it is inconsistent with the above field equation as ℓ is the radius of AdS_2 which is finite. As a result for smooth solutions, A cannot vanish anywhere on M^8 .

Lichnerowicz type theorems for η_{\pm}

The Killing spinors η_{\pm} can be identified with the zero modes of a suitable Dirac-like operator on M^8 . In particular, let us define

$$\mathcal{D}^{(\pm)} = \nabla^{(\pm)} - \mathcal{A}^{(\pm)} , \quad (4.1.14)$$

where $\nabla^{(\pm)} = \nabla + \Psi^{(\pm)}$, ∇ is the Dirac operator on M^8 , and

$$\Psi^{(\pm)} \equiv \Gamma^i \Psi_i^{(\pm)} = \pm \frac{1}{2} A^{-1} \not{A} \mp \frac{1}{4} \not{X} + S + \Gamma_{11} \left(\pm \frac{1}{4} \mathcal{W} - \frac{1}{8} \mathcal{Z} \mp N + \frac{1}{4} \mathcal{P} \right) . \quad (4.1.15)$$

It turns out that if the fields and M^8 satisfy the requirements for the maximum principle to apply, eg M^8 is compact without boundary and all the fields are smooth, then

$$\nabla_i^{(\pm)} \eta_{\pm} = 0 , \quad \mathcal{A}^{(\pm)} \eta_{\pm} = 0 \iff \mathcal{D}^{(\pm)} \eta_{\pm} = 0 . \quad (4.1.16)$$

It is clear that the proof of this in the forward direction is straightforward. To establish the opposite direction for the η_+ spinors, let us assume that $\mathcal{D}^{(+)}\eta_+ = 0$. Then after some extensive algebra using the Bianchi identities and the field equations, one finds [8] that

$$\begin{aligned} \nabla^2 \|\eta_+\|^2 - 2(\partial^i \Phi - A^{-1} \partial^i A) \nabla_i \|\eta_+\|^2 = \\ 2 \|\hat{\nabla}^{(+)} \eta_+\|^2 - (4\kappa + 16\kappa^2) \|\mathcal{A}^{(+)} \eta_+\|^2, \end{aligned} \quad (4.1.17)$$

where

$$\hat{\nabla}_i^{(\pm)} = \nabla_i^{(\pm)} + \kappa \Gamma_i \mathcal{A}^{(\pm)}. \quad (4.1.18)$$

Applying the maximum principle for $\kappa \in (-\frac{1}{4}, 0)$, one concludes that the solutions of the above equation are Killing spinors and that

$$\|\eta_+\| = \text{const}. \quad (4.1.19)$$

Similarly assuming that $\mathcal{D}^{(-)}\eta_- = 0$, one can establish the identity

$$\begin{aligned} \nabla^2 (A^{-2} \|\eta_-\|^2) - 2(\partial^i \Phi - A^{-1} \partial^i A) \nabla_i (A^{-2} \|\eta_-\|^2) = \\ 2A^{-2} \|\hat{\nabla}^{(-)} \eta_-\|^2 - (4\kappa + 16\kappa^2) A^{-2} \|\mathcal{A}^{(-)} \eta_-\|^2. \end{aligned} \quad (4.1.20)$$

Again the application of the maximum principle for $\kappa \in (-\frac{1}{4}, 0)$ gives that η_- is a Killing spinor and that

$$A^{-1} \|\eta_-\| = \text{const}. \quad (4.1.21)$$

The proof for this for near horizon geometries [8] is based on a partial integration argument instead.

4.1.4 Counting of supersymmetries

The counting of supersymmetries for $\text{AdS}_2 \times_w M^8$ backgrounds under the assumptions made in the previous section is a special case of the proof of [8] that IIA horizons always preserve an even number of supersymmetries. Here, we shall briefly repeat the argument. If $N_{\pm} = \dim \text{Ker}(\nabla_i^{(\pm)}, \mathcal{A}^{(\pm)})$, then the number of supersymmetries preserved by the background is $N = N_+ + N_-$. On the other hand from the Lichnerowicz type theorems of the previous section

$$N_{\pm} = \dim \text{Ker} \mathcal{D}^{(\pm)}. \quad (4.1.22)$$

Furthermore, it turns out that $(e^{2\Phi} \Gamma_-)(\mathcal{D}^{(+)})^\dagger = \mathcal{D}^{(-)}(e^{2\Phi} \Gamma_-)$ and so

$$N_- = \dim \text{Ker} \mathcal{D}^{(+)\dagger}. \quad (4.1.23)$$

On the other hand the index of $\mathcal{D}^{(+)}$ is the same as the index of the Dirac operator $\not{\nabla}$ acting on the Majorana representation of $Spin(8)$. The latter vanishes and so $N_+ = N_-$. Thus we conclude that $\text{AdS}_2 \times_w M^8$ solutions preserve

$$N = N_+ + N_- = 2N_- , \quad (4.1.24)$$

supersymmetries confirming that $N = 2k$.

4.2 $\text{AdS}_3 \times_w M^7$

4.2.1 Fields, Bianchi identities and field equations

The fields of AdS_3 backgrounds which are compatible with the AdS_3 symmetries are

$$\begin{aligned} ds^2 &= 2\mathbf{e}^+\mathbf{e}^- + A^2 dz^2 + ds^2(M^7), \\ G &= A\mathbf{e}^+ \wedge \mathbf{e}^- \wedge dz \wedge X + Y, \quad F = F, \\ H &= AW\mathbf{e}^+ \wedge \mathbf{e}^- \wedge dz + Z, \quad S = S, \quad \Phi = \Phi, \end{aligned} \quad (4.2.1)$$

where

$$\begin{aligned} \mathbf{e}^+ &= du, \quad \mathbf{e}^- = (dr + rh), \quad \Delta = 0 \\ h &= -\frac{2}{\ell} dz - 2A^{-1} dA, \end{aligned} \quad (4.2.2)$$

A is the warp factor which depends only on the coordinates of M^7 , (r, u, z) are the coordinates of AdS_3 , X is a 1-form, S, Φ, W are functions, F is a 2-form, Z is a 3-form and Y is a 4-form on M^7 , respectively.

The Bianchi identities of (massive) IIA supergravity can now be rewritten as differential relations of the fields on M^7 as

$$\begin{aligned} dZ &= 0, \quad d(A^3 W) = 0, \quad dS = S d\Phi, \\ dF &= d\Phi \wedge F + SZ + ASW\mathbf{e}^+ \wedge \mathbf{e}^- \wedge dz, \quad dY = d\Phi \wedge Y + Z \wedge F, \\ dX &= -3A^{-1} dA \wedge X + d\Phi \wedge X - WF. \end{aligned} \quad (4.2.3)$$

The Bianchi identity involving dF is consistent if either $S = 0$, or $W = 0$. Therefore there are two distinct AdS_3 backgrounds to consider. One is a standard IIA supergravity background with a non-vanishing component for H on AdS_3 or a massive IIA supergravity background with H that has components only along M^7 .

Decomposing the field equations of (massive) IIA supergravity for the fields (5.1.1), one finds that

$$\begin{aligned} \nabla^2 \Phi &= -3A^{-1} \partial_i A \partial^i \Phi + 2(d\Phi)^2 - \frac{1}{12} Z^2 + \frac{1}{2} W^2 + \frac{5}{4} S^2 + \frac{3}{8} F^2 + \frac{1}{96} Y^2 - \frac{1}{4} X^2, \\ \nabla^k H_{ijk} &= -3A^{-1} \partial^k A H_{ijk} + 2\partial^k \Phi H_{ijk} + \frac{1}{2} Y_{ijk\ell} F^{k\ell} + S F_{ij}, \\ \nabla^j F_{ij} &= -3A^{-1} \partial^j A F_{ij} + \partial^j \Phi F_{ij} - W X_i - \frac{1}{6} Y_{ijk\ell} Z^{jk\ell}, \\ \nabla^i X_i &= \partial_i \Phi X^i - *_7(Z \wedge Y), \\ \nabla^\ell Y_{ijk\ell} &= -3A^{-1} \partial^\ell A Y_{ijk\ell} + \partial^\ell \Phi Y_{ijk\ell} + *_7(Z \wedge X - WY)_{ijk}, \end{aligned} \quad (4.2.4)$$

and that the Einstein equation separates into an AdS component,

$$\nabla^2 \ln A = -\frac{2}{\ell^2} A^{-2} - \frac{3}{\ell^2} A^{-2} (dA)^2 + 2A^{-1} \partial_i A \partial^i \Phi + \frac{1}{2} W^2 + \frac{1}{4} S^2 + \frac{1}{8} F^2 + \frac{1}{96} Y^2 + \frac{1}{4} X^2 \quad (4.2.5)$$

and a transverse component,

$$R_{ij}^{(7)} = 3\nabla_i \nabla_j \ln A + 3A^{-2} \partial_i A \partial_j A + \frac{1}{12} Y_{ij}^2 - \frac{1}{2} X_i X_j - \frac{1}{96} Y^2 \delta_{ij} \\ + \frac{1}{4} X^2 \delta_{ij} - \frac{1}{4} S^2 \delta_{ij} + \frac{1}{4} Z_{ij}^2 + \frac{1}{2} F_{ij}^2 - \frac{1}{8} F^2 \delta_{ij} - 2\nabla_i \nabla_j \Phi , \quad (4.2.6)$$

where ∇ and $R_{ij}^{(7)}$ are the Levi-Civita connection and the Ricci tensor of M^7 , respectively. The latter contracts to

$$R^{(7)} = 3\nabla^2 \ln A + 3A^{-2} (dA)^2 + \frac{1}{4} Z^2 - \frac{7}{4} S^2 - \frac{3}{8} F^2 + \frac{1}{96} Y^2 + \frac{5}{4} X^2 - 2\nabla^2 \Phi \\ = -\frac{6}{\ell^2} A^{-2} - 6A^{-2} (dA)^2 + 12A^{-1} \partial_i A \partial^i \Phi - 4(d\Phi)^2 + \frac{5}{12} Z^2 + \frac{1}{2} W^2 \\ - \frac{7}{2} S^2 - \frac{3}{4} F^2 + \frac{1}{48} Y^2 + \frac{5}{2} X^2 . \quad (4.2.7)$$

This form of the Ricci scalar is essential to establish the maximum principle formulae necessary for identifying the Killing spinors with the zero modes of Dirac-like operators.

4.2.2 Local aspects: solution of KSEs

Solution of KSEs along AdS_3

The gravitino KSE along the AdS_3 directions gives

$$\begin{aligned} \partial_u \epsilon_{\pm} + A^{-1} \Gamma_{+z} (\ell^{-1} - \Xi_{-}) \epsilon_{\mp} &= 0 \\ \partial_r \epsilon_{\pm} - A^{-1} \Gamma_{-z} \Xi_{+} \epsilon_{\mp} &= 0 \\ \partial_z \epsilon_{\pm} - \Xi_{\pm} \epsilon_{\pm} + 2r\ell^{-1} A^{-1} \Gamma_{-z} \Xi_{+} \epsilon_{\mp} &= 0 \end{aligned} \quad (4.2.8)$$

where

$$\Xi_{\pm} = \mp \frac{1}{2\ell} + \frac{1}{2} \not{\partial} A \Gamma_z \pm \frac{1}{4} A W \Gamma_{11} - \frac{1}{8} A S \Gamma_z - \frac{1}{16} A \not{F} \Gamma_z \Gamma_{11} - \frac{1}{192} A \not{Y} \Gamma_z \mp \frac{1}{8} A \not{X} . \quad (4.2.9)$$

As in the AdS_2 case, we integrate these equations along r and u , and then along z . First observe that

$$\Theta_{+} = A^{-1} \Gamma_z \Xi_{+} , \quad \Theta_{-} = A^{-1} \Gamma_z (\Xi_{-} - \ell^{-1}) , \quad (4.2.10)$$

and that

$$\Xi_{\pm} \Gamma_{z+} + \Gamma_{z+} \Xi_{\mp} = 0 , \quad (4.2.11)$$

$$\Xi_{\pm} \Gamma_{z-} + \Gamma_{z-} \Xi_{\mp} = 0 . \quad (4.2.12)$$

Integrating along the r and u coordinates, one finds that the Killing spinor can be expressed as in (4.1.6). To integrate along z first note that the only AdS-AdS integrability condition is

$$(\Xi_{\pm}^2 \pm \ell^{-1} \Xi_{\pm}) \epsilon_{\pm} = 0 . \quad (4.2.13)$$

Using this, one finds that the integration along z yields

$$\eta_{\pm} = \sigma_{\pm} + e^{\mp z/\ell} \tau_{\pm} , \quad (4.2.14)$$

where

$$\Xi_{\pm} \sigma_{\pm} = 0 \quad \Xi_{\pm} \tau_{\pm} = \mp \ell^{-1} \tau_{\pm} , \quad (4.2.15)$$

and σ_{\pm}, τ_{\pm} are 16-component spinors counted over the reals, $\Gamma_{\pm} \sigma_{\pm} = \Gamma_{\pm} \tau_{\pm} = 0$, that depend only on the coordinates of M^7 .

Combining all the above results together, one finds that the solution of the KSEs along AdS_3 can be written as

$$\begin{aligned} \epsilon &= \epsilon_+ + \epsilon_- = \sigma_+ + e^{-\frac{z}{\ell}} \tau_+ + \sigma_- + e^{\frac{z}{\ell}} \tau_- \\ &\quad - \ell^{-1} u A^{-1} \Gamma_{+z} \sigma_- - \ell^{-1} r A^{-1} e^{-\frac{z}{\ell}} \Gamma_{-z} \tau_+ , \end{aligned} \quad (4.2.16)$$

where the dependence of ϵ on the AdS_3 coordinates (u, r, z) is given explicitly while the dependence on the coordinates y of M^7 is via that of σ_{\pm}, τ_{\pm} spinors.

Remaining independent KSEs

As we have seen the KSEs of (massive) IIA supergravity have been solved provided that one imposes the additional conditions (4.2.15). It is convenient to interpret these as new additional KSEs on M^7 . In order to describe simultaneously the conditions on both the σ_{\pm} and τ_{\pm} spinors, we write $\chi_{\pm} = \sigma_{\pm}, \tau_{\pm}$ and introduce

$$\begin{aligned} \mathbb{B}^{(\pm)} &= \mp \frac{c}{2\ell} + \frac{1}{2} \not\partial A \Gamma_z \pm \frac{1}{4} A W \Gamma_{11} - \frac{1}{8} A S \Gamma_z \\ &\quad - \frac{1}{16} A \not{F} \Gamma_z \Gamma_{11} - \frac{1}{192} A \not{Y} \Gamma_z \mp \frac{1}{8} A \not{X} , \end{aligned} \quad (4.2.17)$$

where $c = 1$ when $\chi_{\pm} = \sigma_{\pm}$ and $c = -1$ when $\chi_{\pm} = \tau_{\pm}$.

Using this, the remaining independent KSEs are

$$\nabla_i^{(\pm)} \chi_{\pm} = 0 , \quad \mathcal{A}^{(\pm)} \chi_{\pm} = 0 , \quad \mathbb{B}^{(\pm)} \chi_{\pm} = 0 , \quad (4.2.18)$$

where

$$\begin{aligned} \nabla_i^{(\pm)} &= \nabla_i + \Psi_i^{(\pm)} , \\ \mathcal{A}^{(\pm)} &= \not\partial \Phi + \frac{1}{12} \not{Z} \Gamma_{11} \mp \frac{1}{2} W \Gamma_z \Gamma_{11} \\ &\quad + \frac{5}{4} S + \frac{3}{8} \not{F} \Gamma_{11} + \frac{1}{96} \not{Y} \pm \frac{1}{4} \not{X} \Gamma_z , \end{aligned} \quad (4.2.19)$$

and where

$$\Psi_i^{(\pm)} = \pm \frac{1}{2} A^{-1} \partial_i A + \frac{1}{8} \not{Z}_i \Gamma_{11} + \frac{1}{8} S \Gamma_i + \frac{1}{16} \not{F} \Gamma_i \Gamma_{11} + \frac{1}{192} \not{Y} \Gamma_i \pm \frac{1}{8} \not{X} \Gamma_{zi} . \quad (4.2.20)$$

It is clear that the first two equations in (4.2.18) are the restrictions imposed on χ_{\pm} from gravitino and dilatino KSEs of (massive) IIA supergravity on M^7 , while the last equation has arisen from the integration of the supergravity KSEs on AdS_3 . All the other integrability conditions that arise in the analysis follow from (4.2.18), the Bianchi identities and the field equations.

Counting supersymmetries

The number of supersymmetries preserved by $\text{AdS}_3 \times_w M^7$ backgrounds is the number of solutions of the KSEs (4.2.18). Thus

$$N = N_+ + N_- = (N_{\sigma_+} + N_{\tau_+}) + (N_{\sigma_-} + N_{\tau_-}) , \quad (4.2.21)$$

where $N_{\sigma_{\pm}}$ and $N_{\tau_{\pm}}$ denote the number of σ_{\pm} and τ_{\pm} Killing spinors, respectively. To prove that AdS_3 backgrounds preserve an even number of supersymmetries observe that if χ_- , for $\chi_- = \sigma_-$ or $\chi_- = \tau_-$, is a Killing spinor, ie it solves all the three equations in (4.2.18), then

$$\chi_+ = A^{-1} \Gamma_{+z} \chi_- , \quad (4.2.22)$$

also solves the KSEs (4.2.18). Vice versa if χ_+ solves the KSEs in (4.2.18), then

$$\chi_- = A \Gamma_{-z} \chi_+ , \quad (4.2.23)$$

also solves the KSEs. Therefore $N_+ = N_-$ and so $N = 2N_-$. Observe also that if $N_{\sigma_+}, N_{\tau_+} \neq 0$ or $N_{\sigma_-}, N_{\tau_-} \neq 0$, then $N = 2(N_{\sigma_-} + N_{\tau_-})$.

4.2.3 Global aspects

Here we shall demonstrate that the Killing spinors can be identified with the zero modes of a suitable Dirac-like operator on M^7 . We shall demonstrate this using the Hopf maximum principle as for the case of $\text{AdS}_2 \times_w M^8$ backgrounds. As we have already mentioned the Bianchi identity for F in (B.5.6) implies that there are two different $\text{AdS}_3 \times_w M^7$ backgrounds to consider depending on whether the mass term vanishes and H is allowed to have a component along AdS_3 , or the mass term does not vanish and H has components only along M^7 . Unlike the local analysis we have presented so far, the proof below of the Lichnerowicz type theorems is sensitive to the two different cases and they will be investigated separately. However, the end result is the same including coefficients in some key formulae. Because of this and to save space, we shall present them together in the summary of the proof described below.

Furthermore, an argument similar to the one we have presented for AdS_2 backgrounds implies that for smooth solutions A does not vanish at any point on M^7 . This is based on the investigation of the field equation for A .

Lichnerowicz type theorems for σ_{\pm} and τ_{\pm}

To begin let us introduce the modified parallel transport operator

$$\hat{\nabla}_i^{(+)} = \nabla_i^{(+)} - \frac{1}{7} A^{-1} \Gamma_{iz} \mathbb{B}^{(+)} - \frac{1}{7} \Gamma_i \mathcal{A}^{(+)}, \quad (4.2.24)$$

and the associated Dirac-like operator

$$\mathcal{D}^{(+)} = \hat{\nabla}^{(+)} - A^{-1} \Gamma_z \mathbb{B}^{(+)} - \mathcal{A}^{(+)} . \quad (4.2.25)$$

It is clear that if χ_+ is a Killing spinor, for $\chi_+ = \sigma_+$ or $\chi_+ = \tau_+$, ie satisfies the conditions (4.2.18), then $\mathcal{D}^{(+)}\chi_+ = 0$. To prove the converse suppose that $\mathcal{D}^{(+)}\chi_+ = 0$, then after some computation which utilizes the field equations, Bianchi identities (and has been presented in appendix B.5), one can establish the identity

$$\begin{aligned} \nabla^2 \|\chi_+\|^2 &+ (3A^{-1}\partial_i A - 2\partial_i \Phi)\nabla^i \|\chi_+\|^2 = \|\hat{\nabla}^{(+)}\chi_+\|^2 + \frac{16}{7} \|A^{-1}\Gamma_z \mathbb{B}^{(+)}\chi_+\|^2 \\ &+ \frac{4}{7} \langle A^{-1}\Gamma_z \mathbb{B}^{(+)}\chi_+, \mathcal{A}^{(+)}\chi_+ \rangle + \frac{2}{7} \|\mathcal{A}^{(+)}\chi_+\|^2 . \end{aligned} \quad (4.2.26)$$

First observe that the right-hand-side of the above expression is positive semi-definite. Applying the maximum principle on $\|\chi_+\|^2$, one concludes that $\nabla^{(+)}\chi_+ = \mathbb{B}^{(+)}\chi_+ = \mathcal{A}^{(+)}\chi_+ = 0$ and that

$$\|\chi_+\| = \text{const.} \quad (4.2.27)$$

Therefore χ_+ is a Killing spinor. Thus provided that the fields and M^7 satisfy the conditions for the maximum principle to apply, we have established that

$$\nabla_i^{(+)}\chi_+ = 0, \quad \mathbb{B}^{(+)}\chi_+ = 0, \quad \mathcal{A}^{(+)}\chi_+ = 0 \iff \mathcal{D}^{(+)}\chi_+ = 0. \quad (4.2.28)$$

It is remarkable that the zero modes of $\mathcal{D}^{(+)}$ satisfy all three KSEs.

Although we have presented Lichnerowicz type theorems for σ_+ and τ_+ spinors, there is another similar theorem for σ_- and τ_- spinors. This can be established either by a direct computation or by using (4.2.23) which relates the χ_+ with the χ_- spinors. For this observe that in addition to the KSEs, the Clifford algebra operation $A\Gamma_{-z}$ intertwines between the corresponding Dirac-like operators $\mathcal{D}^{(+)}$ and $\mathcal{D}^{(-)}$.

Counting supersymmetries again

A consequence of the theorems of the previous section is that the number of supersymmetries of $\text{AdS}_3 \times_w M^7$ backgrounds can be counted in terms of the zero modes of the Dirac-like operators $\mathcal{D}^{(\pm)}$. In particular, one has that

$$N = 2(\dim \text{Ker } \mathcal{D}^{(-)}|_{c=1} + \dim \text{Ker } \mathcal{D}^{(-)}|_{c=-1}) . \quad (4.2.29)$$

It is likely that the dimension of these kernels, as the dimension of the Kernel of the standard Dirac operator, depend on the geometry of M^7 , ie they are not topological.

4.3 $\text{AdS}_4 \times_w \mathcal{M}^6$

4.3.1 Fields, Bianchi identities and field equations

The fields of $\text{AdS}_4 \times_w M^6$ backgrounds are

$$ds^2 = 2\mathbf{e}^+ \mathbf{e}^- + A^2(dz^2 + e^{2z/\ell} dx^2) + ds^2(M^6) ,$$

$$\begin{aligned} G &= A^2 e^{z/\ell} e^+ \wedge e^- \wedge dz \wedge dx X + Y, \\ H &= H, \quad F = F, \quad \Phi = \Phi, \quad S = S, \end{aligned} \quad (4.3.1)$$

where A, X, Φ and S are functions, Y is a 4-form, H is a 3-form and F is a 2-form on M^6 , respectively, and

$$\mathbf{e}^+ = du, \quad \mathbf{e}^- = dr + rh, \quad h = -\frac{2}{\ell} dz - 2A^{-1} dA, \quad \Delta = 0. \quad (4.3.2)$$

A is the warp factor. The dependence of the fields on the AdS_4 coordinates (u, r, z, x) is given explicitly, while the dependence of the fields of the coordinates y of M^6 is suppressed.

The Bianchi identities of (massive) IIA supergravity impose the following conditions on the various components of the fields.

$$\begin{aligned} dH &= 0, \quad dS = S d\Phi, \quad dF = d\Phi \wedge F + SH, \\ dY &= d\Phi \wedge Y + H \wedge F, \quad d(A^4 X) = A^4 d\Phi. \end{aligned} \quad (4.3.3)$$

Similarly, the field equations of the fluxes of (massive) IIA supergravity give

$$\begin{aligned} \nabla^2 \Phi &= -4A^{-1} \partial^i A \partial_i \Phi + 2(d\Phi)^2 + \frac{5}{4} S^2 + \frac{3}{8} F^2 - \frac{1}{12} H^2 + \frac{1}{96} Y^2 - \frac{1}{4} X^2, \\ \nabla^k H_{ijk} &= -4A^{-1} \partial^k A H_{ijk} + 2\partial^k \Phi H_{ijk} + S F_{ij} + \frac{1}{2} F^{k\ell} G_{ijk\ell}, \\ \nabla^j F_{ij} &= -4A^{-1} \partial^j A F_{ij} + \partial^j \Phi F_{ij} - \frac{1}{6} F^{j\ell} G_{ijk\ell}, \\ \nabla^\ell Y_{ijk\ell} &= -4A^{-1} \partial^\ell A Y_{ijk\ell} + \partial^\ell \Phi Y_{ijk\ell}, \end{aligned} \quad (4.3.4)$$

and the Einstein equation separates into an AdS component,

$$\begin{aligned} \nabla^2 \ln A &= -3\ell^{-2} A^{-2} - 4A^{-2} (dA)^2 + 2A^{-1} \partial_i A \partial^i \Phi + \frac{1}{96} Y^2 \\ &\quad + \frac{1}{4} X^2 + \frac{1}{4} S^2 + \frac{1}{8} F^2, \end{aligned} \quad (4.3.5)$$

and a component on M^6 ,

$$\begin{aligned} R_{ij}^{(6)} &= 4\nabla_i \nabla_j \ln A + 4A^{-2} \partial_i A \partial_j A + \frac{1}{12} Y_{ij}^2 - \frac{1}{96} Y^2 \delta_{ij} + \frac{1}{4} X^2 \delta_{ij} \\ &\quad - \frac{1}{4} S^2 \delta_{ij} + \frac{1}{4} H_{ij}^2 + \frac{1}{2} F_{ij}^2 - \frac{1}{8} F^2 \delta_{ij} - 2\nabla_i \nabla_j \Phi, \end{aligned} \quad (4.3.6)$$

where $R_{ij}^{(6)}$ is the Ricci tensor of M^6 . The latter contracts to

$$\begin{aligned} R^{(6)} &= 4\nabla^2 \ln A + 4A^{-2} (dA)^2 + \frac{1}{48} Y^2 + \frac{3}{2} X^2 - \frac{3}{2} S^2 + \frac{1}{4} H^2 - \frac{1}{4} F^2 - 2\nabla^2 \Phi \\ &= -12\ell^{-2} A^{-2} - 12A^{-2} (dA)^2 + \frac{1}{24} Y^2 + 3X^2 - 3S^2 + \frac{5}{12} H^2 \\ &\quad - \frac{1}{2} F^2 + 16A^{-1} \partial_i A \partial^i \Phi - 4(d\Phi)^2. \end{aligned} \quad (4.3.7)$$

This expression for the Ricci scalar is used in the proof of the Lichnerowicz type theorems for these backgrounds.

4.3.2 Local aspects: Solution of KSEs

Solution of KSEs on AdS₄

The KSEs of (massive) IIA supergravity along the AdS₄ directions give

$$\begin{aligned}
\partial_u \epsilon_{\pm} + A^{-1} \Gamma_{+z} (\ell^{-1} - \Xi_{\mp}) \epsilon_{\mp} &= 0 , \\
\partial_r \epsilon_{\pm} - A^{-1} \Gamma_{-z} \Xi_{\pm} \epsilon_{\mp} &= 0 , \\
\partial_z \epsilon_{\pm} - \Xi_{\pm} \epsilon_{\pm} + 2r \ell^{-1} A^{-1} \Gamma_{-z} \Xi_{\pm} \epsilon_{\mp} &= 0 , \\
\partial_x \epsilon_{+} + e^{z/\ell} \Gamma_{zx} \Xi_{+} \epsilon_{+} &= 0 , \\
\partial_x \epsilon_{-} + e^{z/\ell} \Gamma_{zx} (\Xi_{-} - \ell^{-1}) \epsilon_{-} &= 0 ,
\end{aligned} \tag{4.3.8}$$

where

$$\Xi_{\pm} = \mp \frac{1}{2\ell} + \frac{1}{2} \not{A} \Gamma_z - \frac{1}{8} A \Gamma_z - \frac{1}{16} A \not{r} \Gamma_z \Gamma_{11} - \frac{1}{192} A \not{Y} \Gamma_z \mp \frac{1}{8} A X \Gamma_x. \tag{4.3.9}$$

Using

$$\begin{aligned}
\Xi_{\pm} \Gamma_{z+} + \Gamma_{z+} \Xi_{\mp} &= 0 , \quad \Xi_{\pm} \Gamma_{z-} + \Gamma_{z-} \Xi_{\mp} = 0 , \\
\Xi_{\pm} \Gamma_{zx} + \Gamma_{zx} \Xi_{\pm} &= \mp \ell^{-1} \Gamma_{zx},
\end{aligned} \tag{4.3.10}$$

one finds that there is only one integrability condition along all AdS₄ directions,

$$(\Xi_{\pm}^2 \pm \ell^{-1} \Xi_{\pm}) \epsilon_{\pm} = 0. \tag{4.3.11}$$

Thus, we can easily integrate the KSEs along AdS₄. In particular, the integration along r, u and z proceeds as for the AdS₃ backgrounds. Then integrating along x , we find that the Killing spinors can be expressed as

$$\begin{aligned}
\epsilon &= \epsilon_{+} + \epsilon_{-} = \sigma_{+} - \ell^{-1} x \Gamma_{xz} \tau_{+} + e^{-\frac{z}{\ell}} \tau_{+} + \sigma_{-} + e^{\frac{z}{\ell}} (\tau_{-} - \ell^{-1} x \Gamma_{xz} \sigma_{-}) \\
&\quad - \ell^{-1} u A^{-1} \Gamma_{+z} \sigma_{-} - \ell^{-1} r A^{-1} e^{-\frac{z}{\ell}} \Gamma_{-z} \tau_{+},
\end{aligned} \tag{4.3.12}$$

where

$$\Xi_{\pm} \sigma_{\pm} = 0 \quad \Xi_{\pm} \tau_{\pm} = \mp \ell^{-1} \tau_{\pm}, \tag{4.3.13}$$

and σ_{\pm} and τ_{\pm} depend only on the coordinates of M^6 . Observe that σ_{\pm} and τ_{\pm} are again 16-component spinors counted over the reals.

Remaining independent KSEs

Having integrated the KSEs of (massive) IIA supergravity along the AdS₄, it remains to identify the remaining independent KSEs. For this, let us collectively denote $(\sigma_{\pm}, \tau_{\pm})$ with χ_{\pm} . It is also convenient to view (4.3.13) as additional KSEs on M^6 . Investigating the various integrability conditions that arise, one finds that the remaining independent KSEs are

$$\nabla_i^{(\pm)} \chi_{\pm} = 0, \quad \mathcal{A}^{(\pm)} \chi_{\pm} = 0, \quad \mathbb{B}^{(\pm)} \chi_{\pm} = 0, \tag{4.3.14}$$

where

$$\begin{aligned}
\nabla_i^{(\pm)} &= \nabla_i + \Psi_i^{(\pm)} , \\
\mathcal{A}^{(\pm)} &= \not\partial\Phi + \frac{1}{12}\not{\Gamma}\Gamma_{11} + \frac{5}{4}S + \frac{3}{8}\not{\Gamma}\Gamma_{11} + \frac{1}{96}\not{Y} \mp \frac{1}{4}X\Gamma_{zx} , \\
\mathbb{B}^{(\pm)} &= \mp \frac{c}{2\ell} + \frac{1}{2}\not{\partial}A\Gamma_z - \frac{1}{8}A\nabla\Gamma_z - \frac{1}{16}A\nabla\Gamma_z\Gamma_{11} \\
&\quad - \frac{1}{192}A\nabla\Gamma_z \mp \frac{1}{8}AX\Gamma_x ,
\end{aligned} \tag{4.3.15}$$

and where

$$\Psi_i^{(\pm)} = \pm \frac{1}{2A}\partial_i A + \frac{1}{8}\not{\Gamma}_i\Gamma_{11} + \frac{1}{8}S\Gamma_i + \frac{1}{16}\not{\Gamma}\Gamma_i\Gamma_{11} + \frac{1}{192}\not{Y}\Gamma_i \mp \frac{1}{8}X\Gamma_{zxi} . \tag{4.3.16}$$

The constant c in $\mathbb{B}^{(\pm)}$ is chosen such that $c = 1$ for $\chi_{\pm} = \sigma_{\pm}$ and $c = -1$ for $\chi_{\pm} = \tau_{\pm}$. Clearly, the first two equations in (4.3.14) arise from the gravitino and dilatino KSEs of (massive) IIA supergravity as adapted on the spinors χ_{\pm} , respectively. The last equation in (4.3.14) implements (4.3.13) on the spinors.

Counting of supersymmetries

The number of Killing spinors of AdS_4 backgrounds is

$$N = N_+ + N_- = (N_{\sigma_+} + N_{\tau_+}) + (N_{\sigma_-} + N_{\tau_-}) , \tag{4.3.17}$$

where $N_{\sigma_{\pm}}$ and $N_{\tau_{\pm}}$ denote the number of σ_{\pm} and τ_{\pm} Killing spinors, respectively.

As for AdS_3 backgrounds one can verify by a direct computation that if χ_- is a Killing spinor, ie solves (4.3.14), then $\chi_+ = A^{-1}\Gamma_{+z}\chi_-$ is also a Killing spinor, and vice-versa if χ_+ is a Killing spinor, then $\chi_- = A\Gamma_{-z}\chi_+$ is also a Killing spinor. Furthermore, one can also verify that if τ_{\pm} is a Killing spinor, then

$$\sigma_{\pm} = \Gamma_{xz}\tau_{\pm} , \tag{4.3.18}$$

is also a Killing spinor, and vice versa if σ_{\pm} is a Killing spinor, then

$$\tau_{\pm} = \Gamma_{xz}\sigma_{\pm} , \tag{4.3.19}$$

is a Killing spinor. As a result of this analysis, $N_{\sigma_+} = N_{\tau_+} = N_{\sigma_-} = N_{\tau_-}$ and so

$$N = 4N_{\sigma_-} . \tag{4.3.20}$$

4.3.3 Global aspects

As in all previous cases, one can demonstrate that if the fields are smooth, then A does not vanish at any point of M^6 . The argument is similar to that presented in the previous two cases and so it will not be repeated here.

Lichnerowicz type theorems for σ_{\pm} and τ_{\pm}

The Killing spinors σ_{\pm} and τ_{\pm} of AdS_4 backgrounds can be identified with the zero modes of a Dirac-like operator on M^6 . To determine this Dirac-like operator first define

$$\hat{\nabla}_i^{(\pm)} = \nabla_i^{(\pm)} - \frac{1}{3}A^{-1}\Gamma_{iz}\mathbb{B}^{(\pm)} - \frac{1}{6}\Gamma_i\mathcal{A}^{(\pm)} . \quad (4.3.21)$$

and the associated Dirac-like operator

$$\mathcal{D}^{(\pm)} \equiv \hat{\nabla}^{(\pm)} = \nabla^{(\pm)} - 2A^{-1}\Gamma_z\mathbb{B}^{(\pm)} - \mathcal{A}^{(\pm)} . \quad (4.3.22)$$

Then one can establish that

$$\nabla_i^{(\pm)}\chi_{\pm} = 0 , \quad \mathbb{B}^{(\pm)}\chi_{\pm} = 0 , \quad \mathcal{A}^{(\pm)}\chi_{\pm} = 0 \iff \mathcal{D}^{(\pm)}\chi_{\pm} = 0 . \quad (4.3.23)$$

It is apparent that if $\chi_{\pm} = (\sigma_{\pm}, \tau_{\pm})$ are Killing spinors, then they are zero modes of $\mathcal{D}^{(\pm)}$. The task is to demonstrate the converse. We shall do this first for χ_+ spinors. In particular let us assume that $\mathcal{D}^{(+)}\chi_+ = 0$. Then after some extensive Clifford algebra calculus which is presented in appendix B.13 and after using the Bianchi identities and the field equations, like (4.3.7), one can show that

$$\begin{aligned} \nabla^2 \|\chi_+\|^2 + (4A^{-1}\partial^i A - 2\partial^i \Phi)\nabla_i \|\chi_+\|^2 &= \|\hat{\nabla}^{(+)}\chi_+\|^2 \\ &+ \frac{16}{3} \|A^{-1}\Gamma_z\mathbb{B}^{(+)}\chi_+\|^2 + \frac{4}{3} \langle A^{-1}\Gamma_z\mathbb{B}^{(+)}\chi_+, \mathcal{A}^{(+)}\chi_+ \rangle \\ &+ \frac{1}{3} \|\mathcal{A}^{(+)}\chi_+\|^2 . \end{aligned} \quad (4.3.24)$$

First observe that the right-hand-side of the above expression is positive semi-definite. Assuming that M^6 and the fields satisfy the requirements for the application of the maximum principle to apply, eg M^6 compact without boundary and fields smooth, one concludes that χ_+ is a Killing spinor and in addition

$$\|\chi_+\| = \text{const} . \quad (4.3.25)$$

This proves (4.3.23) for the χ_+ spinors.

To prove (4.3.23) for the χ_- spinors, one can either perform a similar computation to that of the χ_+ spinors or simply use the relation $\chi_- = A\Gamma_{-z}\chi_+$ between χ_+ and χ_- spinors and observe that the Clifford algebra operation $A\Gamma_{-z}$ intertwines between the Killing spinor equations and the Dirac-like operators. In particular, the analogous maximum principle relation to (4.3.24) for χ_- spinors can be constructed from (4.3.24) by simply setting $\chi_+ = A^{-1}\Gamma_{+z}\chi_-$.

Counting supersymmetries again

A consequence of the theorems of the previous section is that one can count the number of supersymmetries of $\text{AdS}_4 \times_w M^6$ backgrounds in terms of the dimension of the Kernel of $\mathcal{D}^{(\pm)}$ operators. In particular, one has that

$$N = 4 \dim \text{Ker } \mathcal{D}^{(-)}|_{c=1} . \quad (4.3.26)$$

As $\dim \text{Ker } \mathcal{D}^{(-)}|_{c=1} = \dim \text{Ker } \mathcal{D}^{(-)}|_{c=-1} = \dim \text{Ker } \mathcal{D}^{(+)}|_{c=1} = \dim \text{Ker } \mathcal{D}^{(+)}|_{c=-1}$, one can use equivalently in the above formula the dimension of the Kernels of any of these operators.

4.4 $\text{AdS}_n \times_w M^{10-n}$, $n \geq 5$

4.4.1 Fields, Bianchi identities and field equations

For all $\text{AdS}_n \times_w M^{10-n}$, $n \geq 5$, backgrounds, the form fluxes have non-vanishing components only along M^{10-n} . In particular, the fields can be expressed as

$$\begin{aligned} ds^2 &= 2\mathbf{e}^+ \mathbf{e}^- + A^2 \left(dz^2 + e^{2z/\ell} \sum_{a=1}^{n-3} (dx^a)^2 \right) + ds^2(M^{10-n}) , \\ G &= G , \quad H = H , \quad F = F , \quad \Phi = \Phi , \quad S = S , \end{aligned} \quad (4.4.1)$$

where A, Φ and S are functions, G is a 4-form, H is a 3-form and F is a 2-form on M^6 , respectively, and

$$\mathbf{e}^+ = du , \quad \mathbf{e}^- = dr + rh , \quad h = -\frac{2}{\ell} dz - 2A^{-1} dA , \quad \Delta = 0 . \quad (4.4.2)$$

A is the warp factor and ℓ is the radius of AdS_n . The dependence of the fields on the AdS_4 coordinates (u, r, z, x^a) is given explicitly, while the dependence of the fields of the coordinates y of M^{10-n} is suppressed. Clearly additional fluxes will vanish for large enough n , eg AdS_7 backgrounds cannot have 4-form fluxes, $G = 0$.

The Bianchi identities of the (massive) IIA supergravity give

$$\begin{aligned} dH &= 0 , \quad dS = Sd\Phi , \quad dF = d\Phi \wedge F + SH , \\ dG &= d\Phi \wedge G + H \wedge F . \end{aligned} \quad (4.4.3)$$

Furthermore, the field equations of (massive) IIA supergravity give

$$\nabla^2 \Phi = -nA^{-1} \partial^i A \partial_i \Phi + 2(d\Phi)^2 + \frac{5}{4} S^2 + \frac{3}{8} F^2 - \frac{1}{12} H^2 + \frac{1}{96} G^2 , \quad (4.4.4)$$

$$\nabla^k H_{ijk} = -nA^{-1} \partial^k A H_{ijk} + 2\partial^k \Phi H_{ijk} + S F_{ij} + \frac{1}{2} F^{k\ell} G_{ijk\ell} , \quad (4.4.5)$$

$$\nabla^j F_{ij} = -nA^{-1} \partial^j A F_{ij} + \partial^j \Phi F_{ij} - \frac{1}{6} F^{jk\ell} G_{ijk\ell} , \quad (4.4.6)$$

$$\nabla^\ell G_{ijk\ell} = -nA^{-1} \partial^\ell A G_{ijk\ell} + \partial^\ell \Phi G_{ijk\ell} , \quad (4.4.7)$$

and the Einstein equation separates into an AdS component,

$$\nabla^2 \ln A = -(n-1)\ell^{-2} A^{-2} - nA^{-2} (dA)^2 + 2A^{-1} \partial_i A \partial^i \Phi + \frac{1}{96} G^2 + \frac{1}{4} S^2 + \frac{1}{8} F^2 , \quad (4.4.8)$$

and M^{10-n} component,

$$\begin{aligned} R_{ij}^{(10-n)} &= n\nabla_i \nabla_j \ln A + nA^{-2} \partial_i A \partial_j A + \frac{1}{12} G_{ij}^2 - \frac{1}{96} G^2 \delta_{ij} \\ &\quad - \frac{1}{4} S^2 \delta_{ij} + \frac{1}{4} H_{ij}^2 + \frac{1}{2} F_{ij}^2 - \frac{1}{8} F^2 \delta_{ij} - 2\nabla_i \nabla_j \Phi , \end{aligned} \quad (4.4.9)$$

where $R_{ij}^{(10-n)}$ is the Ricci tensor of M^{10-n} . The latter contracts to

$$\begin{aligned}
R^{(10-n)} &= n\nabla^2 \ln A + nA^{-2}(dA)^2 + \frac{n-2}{96}G^2 - \frac{10-n}{4}S^2 + \frac{1}{4}H^2 \\
&\quad + \frac{n-6}{8}F^2 - 2\nabla^2\Phi \\
&= -n(n-1)\ell^{-2}A^{-2} - n(n-1)A^{-2}(dA)^2 + \frac{n-2}{48}G^2 \\
&\quad - \frac{10-n}{2}S^2 + \frac{5}{12}H^2 + \frac{n-6}{4}F^2 \\
&\quad + 4nA^{-1}\partial_i A \partial^i \Phi - 4(d\Phi)^2 .
\end{aligned} \tag{4.4.10}$$

The expression for the Ricci scalar is essential for the proof of the Lichnerowicz type theorems below.

4.4.2 Local aspects: Solution of KSEs

Solution of KSEs along AdS_n

The gravitino KSE of (massive) IIA supergravity along the AdS_n directions gives

$$\begin{aligned}
\partial_u \epsilon_{\pm} + A^{-1}\Gamma_{+z}(\ell^{-1} - \Xi_-)\epsilon_{\mp} &= 0 , \\
\partial_r \epsilon_{\pm} - A^{-1}\Gamma_{-z}\Xi_+\epsilon_{\mp} &= 0 , \\
\partial_z \epsilon_{\pm} - \Xi_{\pm}\epsilon_{\pm} + 2r\ell^{-1}A^{-1}\Gamma_{-z}\Xi_+\epsilon_{\mp} &= 0 , \\
\partial_a \epsilon_+ + e^{z/\ell}\Gamma_{za}\Xi_+\epsilon_+ &= 0 , \\
\partial_a \epsilon_- + e^{z/\ell}\Gamma_{za}(\Xi_- - \ell^{-1})\epsilon_- &= 0 ,
\end{aligned} \tag{4.4.11}$$

where

$$\Xi_{\pm} = \mp \frac{1}{2\ell} + \frac{1}{2}\not\partial A \Gamma_z - \frac{1}{8}A S \Gamma_z - \frac{1}{16}A \not{F} \Gamma_z \Gamma_{11} - \frac{1}{192}A \not{G} \Gamma_z . \tag{4.4.12}$$

Using the identities,

$$\Xi_{\pm}\Gamma_{z+} + \Gamma_{z+}\Xi_{\mp} = 0 , \tag{4.4.13}$$

$$\Xi_{\pm}\Gamma_{z-} + \Gamma_{z-}\Xi_{\mp} = 0 , \tag{4.4.14}$$

$$\Xi_{\pm}\Gamma_{za} + \Gamma_{za}\Xi_{\pm} = \mp \ell^{-1}\Gamma_{za} , \tag{4.4.15}$$

ones finds that all these equations can be solved provided the integrability condition

$$(\Xi_{\pm}^2 \pm \ell^{-1}\Xi_{\pm})\epsilon_{\pm} = 0 , \tag{4.4.16}$$

is satisfied. In particular, one finds that the Killing spinor can be expressed as

$$\begin{aligned}
\epsilon &= \epsilon_+ + \epsilon_- = \sigma_+ - \ell^{-1} \sum_{a=1}^{n-3} x^a \Gamma_{az} \tau_+ + e^{-\frac{z}{\ell}} \tau_+ + \sigma_- + e^{\frac{z}{\ell}} (\tau_- - \ell^{-1} \sum_{a=1}^{n-3} x^a \Gamma_{az} \sigma_-) \\
&\quad - \ell^{-1} u A^{-1} \Gamma_{+z} \sigma_- - \ell^{-1} r A^{-1} e^{-\frac{z}{\ell}} \Gamma_{-z} \tau_+ ,
\end{aligned} \tag{4.4.17}$$

where

$$\Xi_{\pm}\sigma_{\pm} = 0 \quad \Xi_{\pm}\tau_{\pm} = \mp \ell^{-1}\tau_{\pm} , \tag{4.4.18}$$

and σ_{\pm} and τ_{\pm} are 16-component spinors depending only on the coordinates of M^{10-n} . The dependence of the Killing spinors on the AdS_n coordinates is given explicitly while that of the coordinates y of M^{10-n} is via the σ_{\pm} and τ_{\pm} spinors.

Remaining independent KSEs

Having solved the gravitino KSE along AdS_n , $n > 4$, to count the number of supersymmetries preserved by these backgrounds, one has to identify the remaining independent KSEs. There are several integrability conditions which have to be considered. However after using the field equations and the Bianchi identities, one finds that the remaining independent KSEs are

$$\nabla_i^{(\pm)} \chi_{\pm} = 0, \quad \mathcal{A}^{(\pm)} \chi_{\pm} = 0, \quad \mathbb{B}^{(\pm)} \chi_{\pm} = 0, \quad (4.4.19)$$

where

$$\begin{aligned} \nabla_i^{(\pm)} &= \nabla_i + \Psi_i^{(\pm)}, \\ \mathcal{A}^{(\pm)} &= \not{\partial}\Phi + \frac{1}{12} \not{\Gamma}\Gamma_{11} + \frac{5}{4} S + \frac{3}{8} \not{\Gamma}\Gamma_{11} + \frac{1}{96} \not{\Gamma} \\ \mathbb{B}^{(\pm)} &= \mp \frac{c}{2\ell} + \frac{1}{2} \not{\partial}A\Gamma_z - \frac{1}{8} A\Gamma_z - \frac{1}{16} A \not{\Gamma}\Gamma_z\Gamma_{11} - \frac{1}{192} A \not{\Gamma}\Gamma_z, \end{aligned} \quad (4.4.20)$$

and where

$$\Psi_i^{(\pm)} = \pm \frac{1}{2A} \partial_i A + \frac{1}{8} \not{\Gamma}_i \Gamma_{11} + \frac{1}{8} S \Gamma_i + \frac{1}{16} \not{\Gamma}\Gamma_i \Gamma_{11} + \frac{1}{192} \not{\Gamma}\Gamma_i. \quad (4.4.21)$$

We have also set $\chi_{\pm} = (\sigma_{\pm}, \tau_{\pm})$, and $c = 1$ whenever $\chi_{\pm} = \sigma_{\pm}$ and $c = -1$ whenever $\chi_{\pm} = \tau_{\pm}$.

The first two KSEs in (4.4.19) arise from gravitino and dilatino KSEs of (massive) IIA supergravity as they are implemented on χ_{\pm} , respectively. The last equation in (4.4.19) is the condition (4.4.18) which is now interpreted as additional algebraic KSE. All the remaining integrability conditions are implied from (4.4.19), the Bianchi identities and the field equations.

Counting supersymmetries

As in previous cases, the number of supersymmetries N of AdS_n backgrounds is

$$N = N_+ + N_- = (N_{\sigma_+} + N_{\tau_+}) + (N_{\sigma_-} + N_{\tau_-}), \quad (4.4.22)$$

where $N_{\sigma_{\pm}}$ and $N_{\tau_{\pm}}$ denote the number of σ_{\pm} and τ_{\pm} Killing spinors, respectively.

A direct inspection of the remaining independent KSEs (4.4.19) reveals that if χ_- is a solution, then so is $\chi_+ = A^{-1} \Gamma_{+z} \chi_-$, and vice-versa if χ_+ is a Killing spinor, then $\chi_- = A \Gamma_{-z} \chi_+$ is also a Killing spinor. Therefore $N_+ = N_-$. Moreover to count the number of supersymmetries it suffices to count the number of χ_- spinors.

Furthermore if τ_- is a Killing spinor, then $\sigma_- = \Gamma_{az} \tau_-$ is also a Killing spinor, and vice versa if σ_- is a Killing spinor, then $\tau_- = \Gamma_{az} \sigma_-$ is a Killing spinor. Thus $N_{\sigma_-} = N_{\tau_-}$ and so $N = 4N_{\sigma_-}$. Therefore, it remains to count the number of σ_- Killing spinors.

For this observe that if σ_- is a Killing spinor, then

$$\sigma'_- = \Gamma_{ab}\sigma_- , \quad a < b , \quad (4.4.23)$$

is also a Killing spinor. To find N_{σ_-} , one has to count the number of linearly independent $(\sigma, \Gamma_{ab}\sigma_-)$, $a < b$ spinors. This depends on n . For $n = 5$, $a, b = 1, 2$ and $(\sigma, \Gamma_{12}\sigma)$ are linearly independent. Thus AdS_5 backgrounds preserve $N = 8k$ supersymmetries. Next for $n = 6$, $a, b = 1, 2, 3$ and $(\sigma, \Gamma_{12}\sigma, \Gamma_{13}\sigma, \Gamma_{23}\sigma)$ are linearly independent. Thus AdS_6 backgrounds preserve $N = 16k$ supersymmetries. To continue for $n = 7$, $a, b = 1, 2, 3, 4$. It turns out that in this case the Clifford algebra operation Γ_{1234} commutes with all KSEs and therefore one can impose consistently $\Gamma_{1234}\sigma_-^\pm = \pm\sigma_-^\pm$, ie one can restrict σ_- to lie in one of the eigenspaces of Γ_{1234} . In such a case, there are only 4 linearly independent spinors $(\sigma_-, \Gamma_{ab}\sigma_-)$, $a < b$. Thus AdS_7 backgrounds again preserve $16k$ supersymmetries. These results confirm the counting of supersymmetries as stated above.

There are no AdS_n , $n > 7$ backgrounds. This can be seen as follows. If the counting of supersymmetries proceeds in the same way one can show that all such backgrounds preserve 32 supersymmetries. The maximally supersymmetric backgrounds of (massive) IIA supergravity have been classified in [37] and they do not include $\text{AdS}_n \times_w M^{10-n}$ spaces. The same result can be used to rule out the existence of AdS_7 backgrounds that preserve 32 supersymmetries.

4.4.3 Global aspects

Lichnerowicz type theorems for σ_\pm and τ_\pm

As in all previous cases, the Killing spinors χ_\pm of the AdS_n , $n > 4$, backgrounds can be identified with the zero modes of a suitable Dirac-like operator. To prove this first define

$$\hat{\nabla}_i^{(\pm)} = \nabla_i^{(\pm)} - \frac{n-2}{10-n} A^{-1} \Gamma_{iz} \mathbb{B}^{(\pm)} - \frac{1}{10-n} \Gamma_i \mathcal{A}^{(\pm)} , \quad (4.4.24)$$

and

$$\mathcal{D}^{(\pm)} \equiv \hat{\nabla}^{(\pm)} = \nabla^{(\pm)} - (n-2) A^{-1} \Gamma_z \mathbb{B}^{(\pm)} - \mathcal{A}^{(\pm)} . \quad (4.4.25)$$

Then one can show that

$$\nabla^{(\pm)} \chi_\pm = 0 , \quad \mathcal{A}^{(\pm)} \chi_\pm = 0 , \quad \mathbb{B}^{(\pm)} \chi_\pm = 0 \iff \mathcal{D}^{(\pm)} \chi_\pm = 0 . \quad (4.4.26)$$

Clearly the proof of this statement in the forward direction is straightforward. The main task is to prove the converse. It suffices to show this for χ_+ spinors. This is because the Clifford algebra operations $\chi_+ = A^{-1} \Gamma_{+z} \chi_-$ and $\chi_- = A \Gamma_{-z} \chi_+$ which relate these spinors intertwine between the corresponding KSEs and the Dirac-like operators.

Next suppose that χ_+ is a zero mode of the $\mathcal{D}^{(+)}$ operator, $\mathcal{D}^{(+)} \chi_+ = 0$. Then after some computation which is presented in appendix B.7 which involves the use of the field equations

and Bianchi identities, one finds that

$$\begin{aligned} \nabla^2 \|\chi_+\|^2 + (nA^{-1}\partial_i A - 2\partial_i \Phi) \nabla^i \|\chi_+\|^2 &= \|\hat{\nabla} \chi_+\|^2 \\ &+ \frac{16(n-2)}{10-n} \|A^{-1}\Gamma_z \mathbb{B}^{(+)} \chi_+\|^2 + \frac{4(n-2)}{10-n} \langle A^{-1}\Gamma_z \mathbb{B}^{(+)} \chi_+, \mathcal{A}^{(+)} \chi_+ \rangle \\ &+ \frac{2}{10-n} \|\mathcal{A}^{(+)} \chi_+\|^2. \end{aligned} \quad (4.4.27)$$

To proceed one has to solve the above differential equations. For this observe that if the fields are smooth A does not vanish at any point of M^{10-n} . The proof of this is similar to that presented in the previous cases. Furthermore, the right-hand-side of (4.4.27) is positive semi-definite. Thus if M^{10-n} and the fields satisfy the conditions for the application of the maximum principle, eg M^{10-n} compact without boundary and the fields smooth, then the only solution of this is that χ_+ is a Killing spinor and that

$$\|\chi_+\|^2 = \text{const} . \quad (4.4.28)$$

This completes the proof of the theorem.

Counting supersymmetries again

A consequence of the results of the previous section is that the number of supersymmetries of $\text{AdS}_n \times_w M^{10-n}$ backgrounds can be expressed in terms of the dimension of the Kernel of $\mathcal{D}^{(\pm)}$ operators. In particular, one has that

$$N = 4 \dim \text{Ker } \mathcal{D}^{(-)}|_{c=1} . \quad (4.4.29)$$

Equivalently, N can be expressed in terms of $\dim \text{Ker } \mathcal{D}^{(-)}|_{c=-1}$, $\dim \text{Ker } \mathcal{D}^{(+)}|_{c=1}$ and $\dim \text{Ker } \mathcal{D}^{(+)}|_{c=-1}$ as all these numbers are equal. Furthermore $\dim \text{Ker } \mathcal{D}^{(-)}|_{c=1}$ has multiplicity $2^{\lfloor \frac{n}{2} \rfloor - 1}$. This can be seen by an analysis similar to that we have done for the counting the supersymmetries of these backgrounds in section 4.4.2.

4.5 Flux $\mathbb{R}^{n-1,1} \times_w M^{10-n}$ backgrounds

In the limit of large AdS radius ℓ , $\text{AdS}_n \times_w M^{10-n}$ become warped flux $\mathbb{R}^{n-1,1} \times_w M^{10-n}$ backgrounds. Furthermore all the local computations we have performed for $\text{AdS}_n \times_w M^{10-n}$ backgrounds are still valid after taking $\ell \rightarrow \infty$ and so they can be used to investigate the $\mathbb{R}^{n-1,1} \times_w M^{10-n}$ backgrounds. These include the expressions for the fields, Bianchi identities, field equations as well as the local solutions to the KSEs, and the determination of the independent KSEs on M^{10-n} .

However, there are some differences as well. First the counting of supersymmetries is different. This is because the criteria for the linear independence of the solutions of the KSEs on M^{10-n} for $\text{AdS}_n \times_w M^{10-n}$ backgrounds are different from those of $\mathbb{R}^{n-1,1} \times_w M^{10-n}$ backgrounds. Secondly, the global properties of the KSEs for $\text{AdS}_n \times_w M^{10-n}$ and $\mathbb{R}^{n-1,1} \times_w M^{10-n}$

$AdS_n \times_w M^{10-n}$	N
$n = 2$	$2k, k \leq 15$
$n = 3$	$2k, k \leq 15$
$n = 4$	$4k, k \leq 7$
$n = 5$	$8k, k \leq 3$
$n = 6, 7$	16
$n > 7$	—

Table 4.1: The number of supersymmetries N of $AdS_n \times_w M^{10-n}$ backgrounds are given. For $AdS_2 \times_w M^8$, one can show that these backgrounds preserve an even number of supersymmetries provided that M^8 and the fields satisfy the maximum principle. For the counting of supersymmetries of the rest of the backgrounds such an assumption is not necessary. The bounds on k arise from the non-existence of supersymmetric solutions with maximal supersymmetry. For the remaining fractions, it is not known whether there always exist backgrounds preserving the prescribed number of supersymmetries. Supersymmetric AdS_n , $n > 7$, backgrounds do not exist.

backgrounds are different, which originates in differences between the regularity properties of $AdS_n \times_w M^{10-n}$ and $\mathbb{R}^{n-1,1} \times_w M^{10-n}$ backgrounds. It is well known for example that there are no smooth flux compactifications of supergravity theories to $\mathbb{R}^{n-1,1}$ with a compact² internal space M^{10-n} .

4.5.1 Non-existence of flux $\mathbb{R}^{n-1,1} \times_w M^{10-n}$ backgrounds and maximum principle

One of the main properties of AdS backgrounds is that the warp factor A can be no-where vanishing even if M^{10-n} is compact. This is essential for the regularity. As we have seen, this property relies on the radius ℓ of AdS and it is no longer valid in the limit $\ell \rightarrow \infty$.

In fact one can show that the only $\mathbb{R}^{n-1,1} \times_w M^{10-n}$ backgrounds of (massive) IIA supergravity for which the fields and M^{10-n} are chosen such that the maximum principle applies are those for which all fluxes vanish, and the dilaton and warp factor are constant. To see this, observe that the field equation of the warp factor A in all cases can be rewritten as a differential inequality

$$\nabla^2 \ln A + b^i \partial_i \ln A = \Sigma \geq 0, \quad (4.5.1)$$

for some b which depends on A and the dilaton and Σ which depends again on the fields. Therefore it is in a form that the maximum principle can apply. Assuming that the maximum principle applies, the only solution of this equation is that A is constant and $\Sigma = 0$. The latter condition

²We shall demonstrate below that the same conclusion applies under a weaker hypothesis.

in turn gives that all the fluxes must vanish apart from the component of H on M^{10-n} and the dilaton which are not restricted. However the vanishing of the rest of the fields turns the field equation for the dilaton into a maximum principle form. Applying the maximum principle again for this, one finds that the dilaton is constant and the component of H on M^{10-n} vanishes as well. Therefore there are no warped flux $\mathbb{R}^{n-1,1} \times_w M^{10-n}$ backgrounds which satisfy the maximum principle. Observe that this result applies irrespective on whether the solution is supersymmetric or not.

In the context of flux compactifications based on $\mathbb{R}^{n-1,1} \times_w M^{10-n}$ this no-go theorem may be circumvented in various ways. One way is to take M^{10-n} to be non-compact. Another way is to no longer assume that various fields satisfy the properties required for the maximum principle to hold, by weakening the assumption of smoothness. One can also add brane charges which modify the Bianchi identities and the field equations, and/or add higher order corrections. However here we shall focus on the properties of supergravity and we shall simply assume that the fields and M^{10-n} do not satisfy the requirements for maximum principle to apply.

4.5.2 Supersymmetry of flux $\mathbb{R}^{n-1,1} \times_w M^{10-n}$ backgrounds

$\mathbb{R}^{1,1} \times_w M^8$

The proof that $\text{AdS}_2 \times_w M^8$ backgrounds preserve an even number of supersymmetries relies on the maximum principle which is not applicable to $\mathbb{R}^{1,1} \times_w M^8$ supergravity backgrounds. Because of this, we cannot establish in generality that flux $\mathbb{R}^{1,1} \times_w M^8$ backgrounds preserve an even number of supersymmetries. Nevertheless some supersymmetry enhancement is expected. In particular, we have seen that it is a property of (massive) IIA supergravity that if η_- is a Killing spinor then $\eta_+ = \Gamma_+ \Theta_- \eta_-$ is also a Killing spinor. Supersymmetry enhancement takes place whenever $\eta_- \notin \text{Ker } \Theta_-$ and so $\eta_+ \neq 0$. However there is no general argument which leads to $\eta_+ \neq 0$ and so this has to be established on a case by case basis.

The general form of the Killing spinor is

$$\epsilon = \eta_+ + \eta_- + u\Gamma_+ \Theta_- \eta_- + r\Gamma_- \Theta_+ \eta_+ , \quad (4.5.2)$$

for a general choice of η_{\pm} . To establish the above expression from that in (4.1.6) for AdS_2 backgrounds, we have taken the limit $\ell \rightarrow \infty$ and we have used the integrability conditions of the KSEs stated in [8] which read

$$\Gamma_{\mp} \Theta_{\pm} \Gamma_{\pm} \Theta_{\mp} \eta_{\mp} = 0 . \quad (4.5.3)$$

These are automatically satisfied as a consequence of the independent KSEs on M^8 (4.1.8), the Bianchi identities and the field equations. Note that the Killing spinor ϵ is at most linear in the coordinates (u, r) of $\mathbb{R}^{1,1}$. This conclusion arises from the general analysis we have done and it is contrary to the expectation that the Killing spinors of flux $\mathbb{R}^{1,1} \times_w M^8$ backgrounds do not depend on the coordinates of $\mathbb{R}^{1,1}$. Notice also that ϵ does not depend on (u, r) whenever η_{\pm} are in the Kernel of Θ_{\pm} . We shall further comment on these below.

$$\mathbb{R}^{2,1} \times_w M^7$$

The solution of the KSEs (4.2.8) in the limit $\ell \rightarrow \infty$ is

$$\epsilon = \sigma_+ + \sigma_- + u\Gamma_{+z}\Xi_- \sigma_- + r\Gamma_{-z}\Xi_+ \sigma_+ + z(\Xi_+ \sigma_+ + \Xi_- \sigma_-) , \quad (4.5.4)$$

provided that the integrability conditions

$$(\Xi_{\pm})^2 \sigma_{\pm} = 0 , \quad (4.5.5)$$

are satisfied, where σ_{\pm} depend only on the coordinates of M^7 . Moreover necessary and sufficient conditions for ϵ to be a Killing spinor are that σ_{\pm} must satisfy the KSEs (4.2.18) on M^7 .

Comparing the above result with that for $\text{AdS}_3 \times_w M^7$ backgrounds, one notices that the τ_{\pm} spinors do not arise. This is because the τ_{\pm} spinors are not linearly independent from the σ_{\pm} ones for $\mathbb{R}^{2,1} \times_w M^7$ backgrounds. The same applies for the rest of $\mathbb{R}^{n-1,1} \times_w M^{10-n}$ backgrounds and so the explanation will not be repeated below.

To count the number N of supersymmetries preserved by the $\mathbb{R}^{2,1} \times_w M^7$ backgrounds, first observe that $N = N_{\sigma_+} + N_{\sigma_-}$, where N_{σ_+} and N_{σ_-} is the number of σ_+ and σ_- Killing spinors, respectively. Then notice that if σ_- is a Killing spinor, then $\sigma_+ = A^{-1}\Gamma_{+z}\sigma_-$ is also a Killing spinor, and vice versa if σ_+ is a Killing spinor then $\sigma_- = A\Gamma_{-z}\sigma_+$ is also a Killing spinor. Therefore $N_{\sigma_+} = N_{\sigma_-}$, and so $N = 2N_{\sigma_-}$, ie the $\mathbb{R}^{2,1} \times_w M^7$ solutions preserve an even number of supersymmetries confirming the general formula.

$$\mathbb{R}^{3,1} \times_w M^6$$

The solution of the KSEs (4.3.8) in the limit $\ell \rightarrow \infty$ is

$$\epsilon = \sigma_+ + \sigma_- + u\Gamma_{+z}\Xi_- \sigma_- + r\Gamma_{-z}\Xi_+ \sigma_+ + (z + x\Gamma_{xz})(\Xi_+ \sigma_+ + \Xi_- \sigma_-) , \quad (4.5.6)$$

provided that the integrability conditions

$$(\Xi_{\pm})^2 \sigma_{\pm} = 0 , \quad (4.5.7)$$

are satisfied, where σ_{\pm} depend only on the coordinates of M^6 . Moreover necessary and sufficient conditions for ϵ to be a Killing spinor are that σ_{\pm} must satisfy the KSEs (4.3.14) on M^6 .

The number of supersymmetries preserved by the $\mathbb{R}^{3,1} \times_w M^6$ backgrounds is $N = N_{\sigma_+} + N_{\sigma_-}$ where N_{σ_+} and N_{σ_-} is the number of σ_+ and σ_- Killing spinors, respectively. Furthermore as in the $\mathbb{R}^{2,1} \times_w M^7$ case above $N_{\sigma_+} = N_{\sigma_-}$. In addition, if σ_{\pm} is a Killing spinor so is $\sigma'_{\pm} = \Gamma_{zx}\sigma_{\pm}$. As a result $N_{\sigma_{\pm}}$ are even numbers. Thus $\mathbb{R}^{3,1} \times_w M^6$ backgrounds preserve $4k$ supersymmetries.

$$\mathbb{R}^{n-1,1} \times_w M^{10-n} \text{ for } n \geq 5$$

The solution of the KSEs (4.4.11) in the limit $\ell \rightarrow \infty$ is

$$\epsilon = \sigma_+ + \sigma_- + u\Gamma_{+z}\Xi_- \sigma_- + r\Gamma_{-z}\Xi_+ \sigma_+ + (z + \sum_{a=1}^{n-3} x^a \Gamma_{az})(\Xi_+ \sigma_+ + \Xi_- \sigma_-) , \quad (4.5.8)$$

provided that the integrability conditions

$$(\Xi_{\pm})^2 \sigma_{\pm} = 0, \quad (4.5.9)$$

are satisfied, where σ_{\pm} depend only on the coordinates of M^{10-n} . Moreover necessary and sufficient conditions for ϵ to be a Killing spinor are that σ_{\pm} must satisfy the KSEs (4.4.19) on M^{10-n} .

To count the number of supersymmetries preserved by these backgrounds observe that $N = N_{\sigma_+} + N_{\sigma_-}$ and that $N_{\sigma_+} = N_{\sigma_-}$ as in previous cases. Therefore it suffices to count the multiplicity of σ_- Killing spinors. For this notice that for $\mathbb{R}^{n-1,1} \times_w M^{10-n}$ backgrounds, the z coordinate can be treated in the same way as the x^a coordinates. As a result let us denote with $x^{a'} = (z, x^a)$ all the coordinates of $\mathbb{R}^{n-1,1}$ transverse to the lightcone. Furthermore observe that if σ_- is a Killing spinor so is $\Gamma_{a'b'}\sigma_-$ for $a' < b'$. Therefore it suffices to count the linearly independent $(\sigma_-, \Gamma_{a'b'}\sigma_-)$, $a' < b'$ spinors in each case. For the analysis that follows, we shall choose directions for convenience and therefore the analysis is not fully covariant. However, it can be made covariant as that presented in [13].

For $\mathbb{R}^{4,1} \times_w M^5$ a direct computation reveals that there are 4 linearly independent $(\sigma_-, \Gamma_{a'b'}\sigma_-)$, $a' < b'$, $a', b' = 1, 2, 3$, spinors leading to the conclusion that such backgrounds preserve $N = 8k$ supersymmetries.

For $\mathbb{R}^{5,1} \times_w M^4$, one can impose the projection $\Gamma_{1234}\sigma_{\pm}^{\pm} = \pm\sigma_{\pm}^{\pm}$ as $a', b' = 1, 2, 3, 4$ and since Γ_{1234} commutes with all KSEs. If σ_- is chosen to be in one of the two eigenspaces of Γ_{1234} , then only 4 of the $(\sigma_-, \Gamma_{a'b'}\sigma_-)$, $a' < b'$, spinors are linearly independent. As a result, $\mathbb{R}^{5,1} \times_w M^4$ backgrounds preserve $N = 8k$ supersymmetries as well.

A similar argument implies to the counting of supersymmetries for $\mathbb{R}^{6,1} \times_w M^3$ backgrounds. Imposing that σ_- lies in one of the eigenspaces of Γ_{1234} , only 8 of the spinors $(\sigma_-, \Gamma_{a'b'}\sigma_-)$, $a' < b'$, $a', b' = 1, 2, 3, 4, 5$ are linearly independent. Therefore these backgrounds preserve 16k supersymmetries.

For $\mathbb{R}^{7,1} \times_w M^2$ backgrounds, σ_- can be chosen to lie in an eigenspace of two Clifford algebra operators, say Γ_{1234} and Γ_{1256} . In such a case only 8 of the spinors $(\sigma_-, \Gamma_{a'b'}\sigma_-)$, $a' < b'$, $a', b' = 1, 2, 3, 4, 5, 6$ are linearly independent and so such backgrounds also preserve 16k supersymmetries.

Next consider the $\mathbb{R}^{8,1} \times_w M^1$ backgrounds which include the D8-brane solution. In this case σ_- can be chosen to lie in an eigenspace of Γ_{1234} , Γ_{1256} and Γ_{1357} . For such a choice, there are only 8 of the spinors $(\sigma_-, \Gamma_{a'b'}\sigma_-)$, $a' < b'$, $a', b' = 1, 2, 3, 4, 5, 6, 7$ are linearly independent. Therefore such backgrounds also preserve $N = 16k$ supersymmetries.

It should also pointed out that massive IIA supergravity does not have a maximally supersymmetric solution while all the maximally supersymmetric solutions of standard IIA supergravity are locally isometric to $\mathbb{R}^{9,1}$ with vanishing fluxes and constant dilaton [37]. This in particular implies that N is further restricted. The results have been summarized in table 2.

$\mathbb{R}^{n-1,1} \times_w M^{10-n}$	N
$n = 2$	$N < 31$
$n = 3$	$2k, k \leq 15$
$n = 4$	$4k, k \leq 15$
$n = 5$	$8, 16, 24$
$n = 6$	$8, 16, 24$
$n = 7, 8, 9$	16
$n = 10$	32

Table 4.2:

The number of supersymmetries N of $\mathbb{R}^{1,1} \times_w M^{10-n}$ is not a priori an even number. The corresponding statement for AdS_2 backgrounds is proven using global considerations which are not applicable in this case. For the rest, the counting of supersymmetries follows from the properties of KSEs and the classification results of [37, 49]. Furthermore, if the Killing spinors do not depend on $\mathbb{R}^{n-1,1}$ coordinates, then all backgrounds with $N > 16$ are locally isometric to $\mathbb{R}^{9,1}$ with zero fluxes and constant dilaton as a consequence of the homogeneity conjecture [39].

4.6 On the factorization of Killing spinors

4.6.1 AdS backgrounds

Having solved the KSEs of $AdS_n \times_w M^{10-n}$ backgrounds without any assumptions on the form of the Killing spinors, one can address the question of whether the Killing spinors of these spaces factorize as $\epsilon = \psi \otimes \xi$ where ψ is a Killing spinor on AdS_n and ξ is a Killing spinor on M^{10-n} . In particular, ψ is assumed to satisfy a KSE of the type

$$\nabla_\mu \psi + \lambda \gamma_\mu \psi = 0, \quad (4.6.1)$$

where ∇ is the spin connection of AdS_n and λ is a constant related to the radius of AdS_n . This is an assumption which has been extensively used in the literature.

This issue has already been addressed in [13] and [14] for the AdS_n backgrounds of D=11 and IIB supergravities. In particular, it has been found that such a factorization does not occur. In addition if one insists on such a factorization, then one gets the incorrect counting for the supersymmetries of well-known backgrounds like $AdS_5 \times S^5$ and $AdS_7 \times S^4$. The same applies for the backgrounds of (massive) IIA supergravity we have investigated here. After an analysis similar to the one which has been performed in [13] and [14], one finds that the Killing spinors we have found do not factorize into Killing spinors on AdS_n and Killing spinors on M^{10-n} .

4.6.2 Flat backgrounds

The issue of factorization of Killing spinors for $\mathbb{R}^{n-1,1} \times_w M^{10-n}$ backgrounds is closely related to whether the Killing spinors ϵ we have found exhibit a linear dependence on the $\mathbb{R}^{n-1,1}$ co-

ordinates. This is because if the Killing spinors factorize, then they should not depend on the coordinates of $\mathbb{R}^{n-1,1}$ for the chosen coordinate system. As σ_{\pm} must lie in the Kernel of $(\Xi_{\pm})^2$ as a consequence of integrability conditions, the Killing spinors ϵ exhibit a $\mathbb{R}^{n-1,1}$ coordinate dependence, iff $\sigma_{\pm} \notin \text{Ker } \Xi_{\pm}$. In many examples we have investigated, $\sigma_{\pm} \in \text{Ker } (\Xi_{\pm})^2$ implies that $\sigma_{\pm} \in \text{Ker } \Xi_{\pm}$ and so the Killing spinors ϵ do not depend on the coordinates of $\mathbb{R}^{n-1,1}$. However, we have not been able to prove this in general.

Suppose that all Killing spinors do not depend on the coordinates of $\mathbb{R}^{n-1,1}$. If $N > 16$, the homogeneity conjecture [39] applied on the KSEs on M^{10-n} implies that M^{10-n} is homogenous space and all the fields are invariant. In particular, A and Φ are constant. Then the field equations of A and Φ imply that for all such backgrounds the fluxes vanish. As a consequence all such backgrounds with $N > 16$ are locally isometric to $\mathbb{R}^{9,1}$ with zero fluxes and constant dilaton.

4.7 Summary

In this chapter, the Killing spinor equations of $\text{AdS}_n \times_w M^{10-n}$ and $\mathbb{R}^{1,n-1} \times_w M^{10-n}$ IIA backgrounds are solved. As a result, it was possible to determine the supersymmetry fractions preserved by these spaces. AdS_n backgrounds preserve $N = 2\lfloor \frac{n}{2} \rfloor k$ supersymmetries for $n \leq 4$ and $N = 2\lfloor \frac{n}{2} \rfloor + 1 k$ supersymmetries for $4 < n \leq 7$. It was also proven that there are no supersymmetric IIA AdS_n backgrounds for $n \geq 8$. $\mathbb{R}^{1,n-1}$ backgrounds preserve $N = 2\lfloor \frac{n}{2} \rfloor k$ supersymmetries for $2 < n \leq 4$ and $N = 2\lfloor \frac{n+1}{2} \rfloor k$ supersymmetries for $4 < n \leq 8$.

Much like discussions of IIB backgrounds, it is often assumed that the Killing spinors of IIA AdS backgrounds factorize into an AdS Killing spinor and a transverse Killing spinor [42, 30, 50, 29]. Again, however, when this assumption is applied as an ansatz in addition to these results, the allowed supersymmetry fractions are further restricted. This indicates that the Killing spinors do not factorize in general, but only in special cases.

Additionally, for each AdS_n background, a Lichnerowicz-type theorem has been proven. These theorems assume that the transverse space satisfies the requirements of the Hopf maximum principle, which I use to prove that the σ_{+} - and τ_{+} -type Killing spinors are of constant length. Simultaneously, they prove that the Killing spinors are exactly the zero modes of a Dirac-like operator on the transverse space.

Similar results have been found for M-theory backgrounds as well. In [13], it was proven that $\text{AdS}_n \times_w M^{11-n}$ backgrounds preserve $N = 2\lfloor \frac{n}{2} \rfloor k$ supersymmetries for $n \leq 4$ and $N = 2\lfloor \frac{n}{2} \rfloor + 1 k$ supersymmetries for $4 < n \leq 7$, and Lichnerowicz-type theorems like those in this dissertation have been proven for these backgrounds. Similarly, it was proven that $\mathbb{R}^{1,n-1}$ backgrounds preserve $N = 2\lfloor \frac{n}{2} \rfloor k$ supersymmetries for $2 < n \leq 4$ and $N = 2\lfloor \frac{n+1}{2} \rfloor k$ supersymmetries for $4 < n \leq 7$.

M-theory backgrounds, like IIB backgrounds, have been of particular interest in studying the AdS/CFT correspondence [51, 52, 53, 31], because explicit dualities have been found for

$\text{AdS}_4 \times S^7$ and $\text{AdS}_7 \times S^4$ backgrounds. IIA and M-theory backgrounds are closely related, as IIA supergravity is the Kaluza-Klein dimensional reduction of 11-dimensional supergravity [54, 55].

Chapter 5

Heterotic Backgrounds

Among the AdS and flat backgrounds that have been discussed in this dissertation, there are several important differences that make heterotic backgrounds unique. First, it is found that heterotic AdS_n backgrounds cannot exist unless $n = 3$. The additional fields of IIA, IIB, and M-theory backgrounds allow for a broader variety of backgrounds than the dilaton and NS-NS 3-form alone can support. It is also known that there are no heterotic AdS_4 backgrounds, supersymmetric or not, with smooth fields and a compact transverse space [56, 57]. Second, the integrability condition of the gravitino KSE takes an especially simple form for heterotic backgrounds, restricting the fields, rather than the Killing spinors. As a result, a given heterotic background can only support spinors corresponding to one of the two chiral spinor representations of $\text{SO}(2,2)$.

An analysis of heterotic backgrounds at first order in α' , or equivalently heterotic supergravity truncated to two loops [58], has also been included. These backgrounds are characterized by a three-form field strength which is not closed, as well as an α' correction to the Riemann curvature, but are otherwise still quite tractable. They are related to the study of hyper-Kähler manifolds with torsion [59, 60]. There are some qualitative differences between the theory at zeroth order in α' and the theory truncated to two loops [61, 62]. By treating α' as a constant, rather than a perturbative parameter, we avoid imposing the restrictions of string compactifications on these backgrounds artificially. There has also been interest in these backgrounds as they relate to solutions to the Strominger system [63, 64, 65, 66, 67].

5.1 AdS_3 backgrounds with $dH = 0$

The investigation of AdS_3 backgrounds will be separated into two cases depending on whether dH vanishes or not. For the common sector of type II supergravities as well as that for the heterotic string with the standard embedding which leads to the vanishing of the anomaly, one has $dH = 0$. Furthermore $dH = 0$ at zeroth order in the α' expansion in the sigma model

approach to the heterotic string. However, in the latter case $dH \neq 0$ to one and higher loops. For applications to the common sector, it is understood that we consider only one of the two chiral copies of the KSEs.

5.1.1 Fields, Field Equations and Bianchi Identities

The most general metric and NS-NS 3-form flux of warped AdS_3 backgrounds which are invariant under the action of the $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ symmetry algebra of AdS_3 are

$$\begin{aligned} ds^2 &= 2\mathbf{e}^+\mathbf{e}^- + A^2 dz^2 + ds^2(M^7), \\ H &= AXe^+ \wedge e^- \wedge dz + G, \end{aligned} \quad (5.1.1)$$

where we have introduced the frame

$$\mathbf{e}^+ = du \quad \mathbf{e}^- = dr - \frac{2r}{\ell} dz - 2rd \ln A, \quad (5.1.2)$$

u, v , and z are the AdS_3 coordinates, ℓ is the AdS radius, and A is the warp factor. For more details on this parametrization of AdS_3 backgrounds see [19]. Furthermore, one finds that the dilaton, Φ , and the warp factor, A , and G depend only on the M^7 coordinates. In addition X and A , and G are functions, and a 3-form on M^7 , respectively.

The heterotic theory has in addition a 2-form gauge field F with gauge group a subgroup of $E_8 \times E_8$ or $SO(32)/\mathbb{Z}_2$ that is associated with the gauge sector. One way to impose the symmetries of AdS_3 on F is to take F to be the curvature of a connection on M^7 that depends only on the coordinates of M^7 . Alternatively, the gaugino¹ KSE for the backgrounds that we shall be considering implies that F vanishes along the AdS_3 directions and that the Lie derivative of F along the isometries of AdS_3 vanishes as well up to gauge transformations. These in particular imply that the associated Pontryagin forms vanish along AdS_3 and depend only on the coordinates of M^7 . Either results are sufficient for the analysis that will follow.

So far, we have not imposed the Bianchi identity on H and (5.1.1) applies equally to backgrounds regardless on whether dH vanishes or not. However imposing now the Bianchi identity, $dH = 0$, one finds

$$d(A^3 X) = 0, \quad dG = 0. \quad (5.1.3)$$

The field equations for the dilatino and 2-form gauge potential can be expressed as

$$\begin{aligned} \nabla^2 \Phi &= -3A^{-1} \partial_i A \partial^i \Phi + 2(d\Phi)^2 - \frac{1}{12} G^2 + \frac{1}{2} X^2, \\ \nabla^k G_{ijk} &= -3A^{-1} \partial^k A G_{ijk} + 2\partial^k \Phi G_{ijk}, \end{aligned} \quad (5.1.4)$$

where $i, j, k = 1, \dots, 7$. Moreover, the AdS component of the Einstein equation reads

$$\nabla^2 \ln A = -\frac{2}{\ell^2} A^{-2} - 3A^{-2} (dA)^2 + 2A^{-1} \partial_i A \partial^i \Phi + \frac{1}{2} X^2, \quad (5.1.5)$$

¹From now on we assume that the gaugino KSE has the same Killing spinors as the gravitino KSE, see [68] for a justification.

and the M^7 components are

$$R_{ij}^{(7)} = 3\nabla_i \nabla_j \ln A + 3A^{-2} \partial_i A \partial_j A + \frac{1}{4} G_{ik_1 k_2} G_j^{k_1 k_2} - 2\nabla_i \nabla_j \Phi, \quad (5.1.6)$$

where ∇ is the Levi-Civita connection on M^7 and $R_{ij}^{(7)}$ is its Ricci tensor. The Ricci scalar curvature of M^7 can be expressed

$$\begin{aligned} R^{(7)} &= 3\nabla^2 \ln A + 3A^{-2} (dA)^2 + \frac{1}{4} G^2 - 2\nabla^2 \Phi \\ &= -\frac{6}{\ell^2} A^{-2} - 6A^{-2} (dA)^2 + \frac{5}{12} G^2 + \frac{1}{2} X^2 + 12A^{-1} \partial_i A \partial^i \Phi - 4(d\Phi)^2. \end{aligned} \quad (5.1.7)$$

This formula for A constant will be used later in the proof of a Lichnerowicz type theorem.

5.1.2 Solution of KSEs along AdS_3

The heterotic gravitino and dilatino KSEs are

$$\nabla_M \epsilon - \frac{1}{8} \not{H}_M \epsilon = 0 + \mathcal{O}(\alpha'^2), \quad (\not{\partial} \Phi - \frac{1}{12} \not{H}) \epsilon = 0 + \mathcal{O}(\alpha'^2). \quad (5.1.8)$$

Therefore, the form of the two KSEs remains the same up to two and possibly higher loops. The gaugino KSE does not contribute in the investigation of backgrounds with $dH = 0$ and so it not included.

First let us focus on the gravitino KSE. The gravitino KSE along the AdS_3 directions reads

$$\begin{aligned} \partial_u \epsilon_{\pm} + A^{-1} \Gamma_{+z} (\ell^{-1} - \Xi_{-}) \epsilon_{\mp} &= 0, \\ \partial_r \epsilon_{\pm} - A^{-1} \Gamma_{-z} \Xi_{+} \epsilon_{\mp} &= 0, \\ \partial_z \epsilon_{\pm} - \Xi_{\pm} \epsilon_{\pm} + \frac{2r}{\ell} A^{-1} \Gamma_{-z} \Xi_{+} \epsilon_{\mp} &= 0, \end{aligned} \quad (5.1.9)$$

where

$$\Xi_{\pm} = \mp \frac{1}{2\ell} + \frac{1}{2} \not{\partial} A \Gamma_z \mp \frac{1}{4} A X, \quad (5.1.10)$$

and $\Gamma_{\pm} \epsilon_{\pm} = 0$. Furthermore, using the relations

$$\Xi_{\pm} \Gamma_{z+} + \Gamma_{z+} \Xi_{\mp} = 0, \quad \Xi_{\pm} \Gamma_{z-} + \Gamma_{z-} \Xi_{\mp} = 0, \quad (5.1.11)$$

we find that there is only one independent integrability condition

$$\left(\Xi_{\pm}^2 \pm \frac{1}{\ell} \Xi_{\pm} \right) \epsilon_{\pm} = \left(-\frac{1}{4\ell^2} - \frac{1}{4} (dA)^2 \mp \frac{1}{4} A X \not{\partial} A \Gamma_z + \frac{1}{16} A^2 X^2 \right) \epsilon_{\pm} = 0. \quad (5.1.12)$$

As the Clifford algebra operator $\not{\partial} A \Gamma_z$ does not have real eigenvalues, the above integrability condition for $\ell < \infty$ can be satisfied provided that

$$dA = 0, \quad -\frac{1}{4\ell^2} + \frac{1}{16} A^2 X^2 = 0. \quad (5.1.13)$$

Thus the warp factor A is constant. The second equation above also implies that the component X of H along AdS_3 is constant. Furthermore, one can write

$$\Xi_{\pm} = \mp \frac{1 + c_1}{2\ell}, \quad (5.1.14)$$

where $c_1 = \frac{\ell}{2}AX = \pm 1$ as implied by (B.2.1).

The KSEs (5.1.9) can be integrated to find

$$\epsilon = \epsilon_+ + \epsilon_- = \sigma_+ + e^{-\frac{z}{\ell}}\tau_+ + \sigma_- + e^{\frac{z}{\ell}}\tau_- - \ell^{-1}uA^{-1}\Gamma_{+z}\sigma_- - \ell^{-1}rA^{-1}e^{-\frac{z}{\ell}}\Gamma_{-z}\tau_+ , \quad (5.1.15)$$

provided that

$$\Xi_{\pm}\sigma_{\pm} = 0 \quad \Xi_{\pm}\tau_{\pm} = \mp \frac{1}{\ell}\tau_{\pm} . \quad (5.1.16)$$

It is understood that the dependence of ϵ on the AdS_3 coordinates is given explicitly while τ_{\pm} and σ_{\pm} depend only on the coordinates of M^7 .

It is clear from (5.1.16) that there are two solutions to the above conditions. If $c_1 = 1$, (5.1.16) implies that $\sigma_{\pm} = 0$. In turn the Killing spinor is

$$\epsilon = \epsilon_+ + \epsilon_- = e^{-\frac{z}{\ell}}\tau_+ + e^{\frac{z}{\ell}}\tau_- - \ell^{-1}rA^{-1}e^{-\frac{z}{\ell}}\Gamma_{-z}\tau_+ . \quad (5.1.17)$$

Alternatively if $c_1 = -1$, (5.1.16) gives $\tau_{\pm} = 0$ and the Killing spinor is

$$\epsilon = \epsilon_+ + \epsilon_- = \sigma_+ + \sigma_- - \ell^{-1}uA^{-1}\Gamma_{+z}\sigma_- . \quad (5.1.18)$$

Therefore depending on the sign of AX , which coincides in the sign of the contribution volume form of AdS_3 in H , there are two distinct cases to consider.

In order to interpret the two cases that arise, note that AdS_3 can be identified, up to a discrete identification, with the group manifold $SL(2, \mathbb{R})$. As such it is parallelizable with respect to either left or right actions of $SL(2, \mathbb{R})$. The two associated connections differ by the sign of their torsion term which in turn is given by the structure constants of the $\mathfrak{sl}(2, \mathbb{R})$. Of course the associated 3-form coincides with the bi-invariant volume form of AdS_3 .

To treat both cases symmetrically, we introduce $\mathbb{B}^{(\pm)}$ which is equal to Ξ_{\pm} when it acts on σ_{\pm} and equal to $\Xi_{\pm} \pm \frac{1}{\ell}$ when it acts on τ_{\pm} . The integrability conditions are then succinctly expressed as $\mathbb{B}^{(\pm)}\chi_{\pm} = 0$, $\chi_{\pm} = \sigma_{\pm}, \tau_{\pm}$, where

$$\mathbb{B}^{(\pm)} = \mp \frac{c_1 + c_2}{2\ell} , \quad (5.1.19)$$

with $c_2 = 1$ when $\chi_{\pm} = \sigma_{\pm}$ and with $c_2 = -1$ when $\chi_{\pm} = \tau_{\pm}$.

The remaining KSEs on M^7 can now be expressed as

$$\nabla_i^{(\pm)}\chi_{\pm} = 0, \quad \mathcal{A}^{(\pm)}\chi_{\pm} = 0, \quad \mathbb{B}^{(\pm)}\chi_{\pm} = 0 , \quad (5.1.20)$$

where

$$\nabla_i^{(\pm)} = \nabla_i + \Psi_i^{(\pm)} , \quad \Psi_i^{(\pm)} = -\frac{1}{8}\mathcal{G}_i , \quad (5.1.21)$$

is a metric connection with skew-symmetric torsion G associated with the gravitino KSE, and

$$\mathcal{A}^{(\pm)} = \not{\partial}\Phi \pm \frac{c_1}{\ell}A^{-1}\Gamma_z - \frac{1}{12}\mathcal{G} , \quad (5.1.22)$$

is associated with the dilatino KSE, and $\mathbb{B}^{(\pm)}$ should be thought as a projector which restricts the first two equations on either σ_{\pm} or τ_{\pm} spinors.

For the investigation of the geometry of these backgrounds it suffices to consider only the τ_+ or the σ_+ spinors. This is because, if χ_- is a solution to the above KSEs, then $\chi_+ = A^{-1}\Gamma_{+z}\chi_-$ is also a solution, and vice versa, if χ_+ is a solution, then $\chi_- = A\Gamma_{-z}\chi_+$ is also a solution. Incidentally, this also implies that the number of supersymmetries preserved by AdS_3 backgrounds is always even. Furthermore, it suffices to investigate the geometry of these backgrounds as described by the σ_+ spinors. As we have mentioned, the τ_+ spinors arise on choosing the other parallelization for AdS_3 and it can be treated symmetrically, see also appendix D.

5.1.3 Geometry

If the solution of the KSEs is determined by the σ_{\pm} spinors, the investigation of the geometry of M^7 can be done as a special case of that of heterotic horizons in [20] which utilized the classification results of [68]. To see this first note that $h = -\frac{2}{\ell}dz$ and so the constant k which enters in the description of geometry for the heterotic horizons is

$$k^2 = h^2 = 4A^{-2}\ell^{-2} . \quad (5.1.23)$$

Next observe that as σ_+ and σ_- are linearly independent, there are two Killing spinors given by

$$\epsilon^1 = \sigma_+ , \quad \epsilon^2 = \sigma_- - \ell^{-1}uA^{-1}\Gamma_{+z}\sigma_- . \quad (5.1.24)$$

Setting now $\sigma_- = A\Gamma_{-z}\sigma_+$ and after rescaling the second spinor with the non-vanishing constant $-2\ell^{-1}A^{-2}$, we find that the two spinors can be rewritten as

$$\epsilon^1 = \sigma_+ , \quad \epsilon^2 = -k^2u\sigma_+ + \Gamma_- \not{h}\sigma_+ . \quad (5.1.25)$$

These are precisely the spinors that appear in the context of heterotic horizons, see [20] for a detailed description of the geometry of M^7 including the emergence of the (left) $\mathfrak{sl}(2, \mathbb{R})$ symmetry of AdS_3 backgrounds as generated by the 1-form Killing spinor bi-linears. Briefly, M^7 admits a G_2 structure compatible with a metric connection $\hat{\nabla}$ with skew-symmetric torsion, $\hat{\nabla}_i X^j = \nabla_i X^j + \frac{1}{2}G^j{}_{ik}X^k$. Furthermore all field equations and KSEs are implied provided [20] that

$$d(e^{2\Phi} \star_7 \varphi) = 0 , \quad dG = 0 , \quad (5.1.26)$$

where

$$G = k\varphi + e^{2\Phi} \star_7 d(e^{-2\Phi}\varphi) , \quad (5.1.27)$$

and φ is the fundamental form of the G_2 structure. The first condition in (5.1.26) is required for the existence of a G_2 structure on M^7 compatible with a metric connection with skew-symmetric torsion [69] and the second condition is the Bianchi identity (5.1.3). The dilatino KSE implies

two conditions, one of which is that the G_2 structure on M^7 must be conformally balanced, $\theta_\varphi = 2d\Phi$, both of which have been incorporated in the expression for G , where θ_φ is the Lee form of φ . The conditions (5.1.26) are simpler than those that have appeared for heterotic horizons, because for AdS_3 backgrounds $dh = 0$.

5.1.4 Geometry of AdS_3 backgrounds with extended supersymmetry

We have shown that AdS_3 backgrounds always preserve an even number of supersymmetries. Furthermore, from the counting of supersymmetries for heterotic horizons [20], one concludes that AdS_3 backgrounds preserve 2, 4, 6 and 8 supersymmetries. In addition, AdS_3 backgrounds that preserve 8 supersymmetries and for which M^7 is compact are locally isometric to either $\text{AdS}_3 \times S^3 \times T^4$ or to $\text{AdS}_3 \times S^3 \times K_3$. Again we shall not give the details of the proof for these results. However, we shall state the key formulae that arise in the investigation of the geometry for each case as they have some differences from those of the heterotic horizons.

Four supersymmetries

Let us first consider the AdS_3 backgrounds with 4 supersymmetries. The two additional spinors can be written as

$$\epsilon^3 = \sigma_+^2, \quad \epsilon^4 = -k^2 u \sigma_+^2 + \Gamma_- \not{h} \sigma_+^2, \quad (5.1.28)$$

where σ_+^2 is linearly independent from $\sigma_+^1 = \sigma_+$ in (5.1.25). In fact it can be shown that the normal form for these spinors up to the action of $\text{Spin}(7)$ can be chosen as $\sigma_+^1 = 1 + e_{1234}$ and $\sigma_+^2 = i(1 - e_{1234})$. The isotropy group of all four spinors is $SU(3)$. Therefore M^7 is a Riemannian manifold equipped with metric $ds_{(7)}^2$ and a 3-form G . Furthermore, the metric connection $\hat{\nabla}$ with skew-symmetric torsion G is compatible with an $SU(3)$ structure. The KSEs restrict this structure on M^7 further. In particular, the $SU(3)$ structure on M^7 is associated with 1-form ξ , 2-form ω , and (3,0)-form χ spinor bilinears such that

$$i_\xi \omega = 0, \quad \mathcal{L}_\xi \omega = 0, \quad i_\xi \chi = 0, \quad \mathcal{L}_\xi \chi = ik\chi, \quad (5.1.29)$$

where ω and χ are the fundamental forms of an $SU(3)$ structure in the directions transverse to ξ . All these forms are $\hat{\nabla}$ -parallel, $\hat{\nabla}\xi = \hat{\nabla}\omega = \hat{\nabla}\chi = 0$. In particular $\hat{\nabla}\xi = 0$ implies that ξ is Killing and that $i_\xi G = k^{-1}dw$, where $w(\xi) = k$. As G is closed $\mathcal{L}_\xi G = 0$. The dilaton Φ is also invariant under ξ . The full set of conditions on ξ , ω and χ can be found in [20].

The solution of these conditions² implies that M^7 can be locally constructed as a circle fibration on a conformally balanced³, $\theta_\omega = 2d\Phi$, KT manifold B^6 with Hermitian form ω , where

²In fact with the data provided M^7 admits a normal almost contact structure which however is further restricted.

³ θ_ω is the Lee form of B^6 .

the tangent space of the circle fibre is spanned by ξ . The canonical bundle of B^6 admits a connection⁴ $\lambda = k^{-1}w$, such that

$$dw^{(2,0)} = 0, \quad dw_{ij}\omega^{ij} = -2k^2, \quad (5.1.30)$$

$i, j = 1, 2, \dots, 6$, i.e. the canonical bundle is holomorphic and the connection satisfies the Hermitian-Einstein instanton condition, and in addition

$$\hat{\rho}_{(6)} = dw, \quad k^{-2}dw \wedge dw + dG_{(6)} = 0, \quad (5.1.31)$$

where

$$\hat{\rho}_{ij}^{(6)} = \frac{1}{2}\hat{R}_{ij}^{(6)}{}^k{}_m I^m{}_k, \quad (5.1.32)$$

is the curvature of the canonical bundle induced from the connection with torsion $G_{(6)} = -i_I d\omega$ on B^6 , and I is the complex structure of B^6 . The first condition is required for M^7 to admit an $SU(3)$ structure compatible with the connection with skew-symmetric torsion G and the second condition is required by the Bianchi identity (5.1.3). Note that B^6 has a $U(3)$ rather than an $SU(3)$ structure compatible with a connection with skew-symmetric torsion. This is because the $(3,0)$ -form χ is not invariant under the action of ξ (5.1.29).

The metric and torsion on M^7 are given from those of B^6 as

$$ds^2(M^7) = k^{-2}w^2 + ds^2(B^6), \quad G = k^{-2}w \wedge dw + G_{(6)}. \quad (5.1.33)$$

This summarizes the geometry for the AdS_3 backgrounds preserving four supersymmetries. Solutions can be constructed using the techniques developed in [20] to find solutions for heterotic horizons.

Six supersymmetries

Next let us turn to AdS_3 backgrounds preserving 6 supersymmetries. For these M^7 admits an $SU(2)$ structure compatible with $\hat{\nabla}$. Furthermore, M^7 can be constructed locally as a $SU(2) = S^3$ fibration over a 4-dimensional manifold B^4 whose self-dual part of the Weyl tensor vanishes. $SU(2)$ twists over B^4 with respect to a (principal bundle) connection λ which has curvature $\mathcal{F}^{r'}$ such that the self-dual part satisfies

$$(\mathcal{F}^{\text{sd}})^{r'} = \frac{k}{4}\omega^{r'}, \quad (5.1.34)$$

where $\omega^{r'}$ are the almost Hermitian forms of a quaternionic Kähler structure on B^4 . The anti-self dual part of \mathcal{F} , \mathcal{F}^{ad} , is not restricted by the KSEs. The dilaton depends only on the coordinates of B^4 . The metric and G on M^7 are given by

$$ds^2(M^7) = \delta_{r's'}\lambda^{r'}\lambda^{s'} + e^{2\Phi}ds^2(B^4), \quad G = CS(\lambda) - \star de^{2\Phi}, \quad (5.1.35)$$

⁴There some differences in the notation of this paper with that of [20]. For example w is denoted in [20] with ℓ . We have made this change because here we have denoted by ℓ the radius of AdS .

where CS is the Chern-Simons⁵ form of λ . The only condition that remains to be solved to find solutions is

$$\mathring{\nabla}^2 e^{2\Phi} = -\frac{1}{2}(\mathcal{F}^{\text{ad}})^2 + \frac{3}{8}k^2 e^{4\Phi}, \quad (5.1.36)$$

where the inner products are taken with respect to the $d\hat{s}^2$ metric. For more details on the geometry of such backgrounds see [20].

Eight supersymmetries

Next let us turn to the AdS_3 backgrounds preserving 8 supersymmetries. The description of the geometry is as that of the backgrounds above preserving 6 supersymmetries. The only differences are that B^4 must be a hyper-Kähler manifold with respect to the $d\hat{s}^2(B^4)$ metric, and that $\mathcal{F}^{\text{sd}} = 0$. The metric and 3-form G of M^7 are given as in (5.1.35) but now we have that

$$\mathring{\nabla}^2 e^{2\Phi} = -\frac{1}{2}(\mathcal{F}^{\text{ad}})^2, \quad (5.1.37)$$

instead of (5.1.36). If B^4 is compact, a partial integration argument reveals that $\mathcal{F}^{\text{ad}} = 0$ and so the only regular solutions, up to discrete identifications, are $\text{AdS}_3 \times S^3 \times K_3$ and $\text{AdS}_3 \times S^3 \times T^4$. If B^4 is not compact, there are many smooth solutions, see [70].

5.1.5 Lichnerowicz type theorem on σ_+ , τ_+

The Killing spinors of AdS_3 backgrounds (5.1.20) can be identified with the zero modes of a suitable Dirac-like operator coupled to fluxes on M^7 , and vice versa. This provides a new example of a Lichnerowicz type theorem for connections whose holonomy is not in a Spin group. This result is analogous to others that have been established for AdS backgrounds in 11-dimensional and type II supergravities [13, 14, 15]. However, there are some differences. One is that the spinor representation in the heterotic case is different from that of the previous mentioned theories. There are also some subtle issues associated with the modification of the Lichnerowicz type of theorem in the presence of α' corrections, which we shall consider in further detail in the next section.

To begin, let us first suppress the α' corrections, and take $dH = 0$. The Lichnerowicz type of theorem with α' corrections will be investigated later. We define the modified gravitino Killing spinor operator,

$$\hat{\nabla}_i^{(+,q_1,q_2)} = \nabla_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2)}, \quad (5.1.38)$$

on the χ_+ spinors, where

$$\mathbb{A}^{(+,q_1,q_2)} = -q_1 A^{-1} \Gamma_z \mathbb{B}^{(+)} + q_2 \mathcal{A}^{(+)}, \quad (5.1.39)$$

⁵Note that if $\mathcal{F} = 0$, $CS(\lambda)$ is proportional to the volume of S^3 .

for some $q_1, q_2 \in \mathbb{R}$. Observe that for $q_1, q_2 \neq 0$, the holonomy of $\hat{\nabla}^{(+, q_1, q_2)}$ is not in $Spin(7)$. Next define the modified Dirac-like operator

$$\mathcal{D}^{(+)} \equiv \Gamma^i \hat{\nabla}_i^{(+, q_1, q_2)} = \Gamma^i \nabla_i + \Gamma^i \Psi_i^{(+)} + 7\mathbb{A}^{(+, q_1, q_2)}. \quad (5.1.40)$$

It is clear that if χ_+ is a Killing spinor, ie satisfies (5.1.20), then it is a zero mode of $\mathcal{D}^{(+)}$. Here we that prove the converse. In particular, we shall show that there is a choice of q_1, q_2 such that all the zero modes of $\mathcal{D}^{(+)}$ are Killing spinors. Thus we shall establish

$$\nabla_i^{(+)} \chi_{\pm} = 0, \quad \mathcal{A}^{(+)} \chi_{\pm} = 0, \quad \mathbb{B}^{(+)} \chi_{\pm} = 0 \iff \mathcal{D}^{(+)} \chi_{+} = 0. \quad (5.1.41)$$

The proof relies on global properties of M^7 , which we assume to be smooth, and compact without boundary.

To prove the theorem, let us assume that $\mathcal{D}^{(+)} \chi_{+} = 0$ and consider the identity

$$\nabla^2 \|\chi_{+}\|^2 = 2\|\nabla \chi_{+}\|^2 + 2\langle \chi_{+}, \nabla^2 \chi_{+} \rangle. \quad (5.1.42)$$

The first term on the right hand side can be further rewritten in terms of the differential operator $\hat{\nabla}^{(+, q_1, q_2)}$ by completing the square as

$$\begin{aligned} 2\|\nabla \chi_{+}\|^2 &= 2\left\| \hat{\nabla}^{(+, q_1, q_2)} \chi_{+} \right\|^2 - 4\left\langle \chi_{+}, \left(\Psi^{(+)} i^{\dagger} + \mathbb{A}^{(+, q_1, q_2)} i^{\dagger} \Gamma^i \right) \nabla_i \chi_{+} \right\rangle \\ &\quad - 2\left\langle \chi_{+}, \left(\Psi^{(+)} i^{\dagger} + \mathbb{A}^{(+, q_1, q_2)} i^{\dagger} \Gamma^i \right) \left(\Psi_i^{(+)} + \Gamma_i \mathbb{A}^{(+, q_1, q_2)} \right) \chi_{+} \right\rangle \\ &= 2\left\| \hat{\nabla}^{(+, q_1, q_2)} \chi_{+} \right\|^2 - 4\left\langle \chi_{+}, \Psi^{(+)} i^{\dagger} \nabla_i \chi_{+} \right\rangle \\ &\quad - 2\left\langle \chi_{+}, \left(\Psi^{(+)} i^{\dagger} - \mathbb{A}^{(+, q_1, q_2)} i^{\dagger} \Gamma^i \right) \left(\Psi_i^{(+)} + \Gamma_i \mathbb{A}^{(+, q_1, q_2)} \right) \chi_{+} \right\rangle, \end{aligned} \quad (5.1.43)$$

while the second term can be rewritten using the identity $\nabla^2 = \nabla^2 - \frac{1}{4}R^{(7)}$, and $\mathcal{D}^{(+)} \chi_{+} = 0$, as

$$\begin{aligned} 2\langle \chi_{+}, \nabla^2 \chi_{+} \rangle &= 2\langle \chi_{+}, \Gamma^i \nabla_i (\Gamma^j \nabla_j \chi_{+}) \rangle + \frac{1}{2} R^{(7)} \|\chi_{+}\|^2 \\ &= \frac{1}{2} R^{(7)} \|\chi_{+}\|^2 - 2\left\langle \chi_{+}, \nabla_i \left(\Gamma^i \Gamma^j \Psi_j^{(+)} + 7\Gamma^i \mathbb{A}^{(+, q_1, q_2)} \right) \chi_{+} \right\rangle \\ &\quad - 2\left\langle \chi_{+}, \left(\Gamma^i \Gamma^j \Psi_j^{(+)} + 7\Gamma^i \mathbb{A}^{(+, q_1, q_2)} \right) \nabla_i \chi_{+} \right\rangle. \end{aligned} \quad (5.1.44)$$

Combining these, $\nabla^2 \|\chi_{+}\|^2$ can be rewritten as,

$$\begin{aligned} \nabla^2 \|\chi_{+}\|^2 &= 2\left\| \hat{\nabla}^{(+, q_1, q_2)} \chi_{+} \right\|^2 + \frac{1}{2} R^{(7)} \|\chi_{+}\|^2 \\ &\quad + \left\langle \chi_{+}, \left[-4\Psi^{(+)} i^{\dagger} - 2\Gamma^i \Gamma^j \Psi_j^{(+)} - 14q_1 A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} - 14q_2 \Gamma^i \mathcal{A}^{(+)} \right] \nabla_i \chi_{+} \right\rangle \\ &\quad + \left\langle \chi_{+}, -2\left(\Psi^{(+)} i^{\dagger} - \mathbb{A}^{(+, q_1, q_2)} i^{\dagger} \Gamma^i \right) \left(\Psi_i^{(+)} + \Gamma_i \mathbb{A}^{(+, q_1, q_2)} \right) \chi_{+} \right\rangle \\ &\quad + \left\langle \chi_{+}, \nabla_i \left[-2\Gamma^i \Gamma^j \Psi_j^{(+)} - 14q_1 A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} - 14q_2 \Gamma^i \mathcal{A}^{(+)} \right] \chi_{+} \right\rangle \end{aligned} \quad (5.1.45)$$

where

$$\Psi_i^{(+)\dagger} = \frac{1}{8} \mathcal{G}_i, \quad \mathbb{B}^{(+)\dagger} = -\frac{c_1 + c_2}{2\ell}, \quad \mathcal{A}^{(+)\dagger} = \not{\partial} \Phi + \frac{c_1}{\ell} A^{-1} \Gamma_z + \frac{1}{12} \mathcal{G}. \quad (5.1.46)$$

Of the terms on the right hand side of (B.8.6), the first term is proportional to the gravitino Killing spinor equation squared, and so we expect that the remaining terms will be equal to some combination of the algebraic KSEs. The third term includes a derivative of χ_+ , however, and so we will attempt to write it in the form

$$\alpha^i \nabla_i \|\chi_+\|^2 + \langle \chi_+, \mathcal{F} \Gamma^i \nabla_i \chi_+ \rangle = \alpha^i \nabla_i \|\chi_+\|^2 - \langle \chi_+, \mathcal{F} (\Gamma^i \Psi_i^{(+)} + 7\mathbb{A}^{(+,q_1,q_2)}) \chi_+ \rangle, \quad (5.1.47)$$

for some vector α and Clifford algebra element \mathcal{F} that depend on the fields. In terms of the fields, the third term in the right hand side of (B.8.6) can be rewritten as

$$\begin{aligned} & \langle \chi_+, \left[-4\Psi^{(+)\dagger} - 2\Gamma^i \Gamma^j \Psi_j^{(+)} - 14q_1 A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} - 14q_2 \Gamma^i \mathcal{A}^{(+)} \right] \nabla_i \chi_+ \rangle \\ &= \langle \chi_+, \left[\frac{7}{\ell} A^{-1} \Gamma^{zi} (q_1 c_1 + q_1 c_2 + 2q_2 c_1) - 14q_2 \Gamma^i \not\partial \Phi \right. \\ & \quad \left. + \frac{1+14q_2}{4} \not\partial \mathcal{G}^i + \frac{3+14q_2}{12} \Gamma \not\partial \mathcal{G}^i \right] \nabla_i \chi_+ \rangle. \end{aligned} \quad (5.1.48)$$

Thus, we find that it can be separated as outlined above if and only if $q_2 = -\frac{1}{7}$. We will use this value of q_2 from here on. Then we find that

$$\begin{aligned} & \langle \chi_+, \left[-4\Psi^{(+)\dagger} - 2\Gamma^i \Gamma^j \Psi_j^{(+)} - 14q_1 A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} + 2\Gamma^i \mathcal{A} \right] \nabla_i \chi_+ \rangle \\ &= \langle \chi_+, \left[\frac{1}{\ell} A^{-1} \Gamma^{zi} (7q_1 c_1 + 7q_1 c_2 - 2c_1) + 2\Gamma^i \not\partial \Phi \right. \\ & \quad \left. - \frac{1}{4} \not\partial \mathcal{G}^i + \frac{1}{12} \Gamma \not\partial \mathcal{G}^i \right] \nabla_i \chi_+ \rangle, \end{aligned} \quad (5.1.49)$$

and so, factoring out a Γ^i on the right,

$$\mathcal{F} = \frac{1}{\ell} A^{-1} \Gamma_z (7q_1 c_1 + 7q_1 c_2 - 2c_1) - 2\not\partial \Phi - \frac{1}{12} \not\partial \mathcal{G}, \quad (5.1.50)$$

and $\alpha_i = 2\partial_i \Phi$.

The \mathcal{F} term part of the third term of (B.8.6) can be combined with the fourth term of (B.8.6) to give

$$\begin{aligned} & \langle \chi_+, -2(\Psi^{(+)\dagger} + q_1 A^{-1} \mathbb{B}^{(+)\dagger} \Gamma^{zi} + \frac{1}{7} \mathcal{A}^{(+)\dagger} \Gamma^i + \frac{1}{2} \mathcal{F} \Gamma^i) \\ & \quad (\Psi_i^{(+)} + q_1 A^{-1} \Gamma_{zi} \mathbb{B}^{(+)} - \frac{1}{7} \Gamma_i \mathcal{A}^{(+)}) \chi_+ \rangle \\ &= \langle \chi_+, -2 \left[\frac{1}{\ell} (3q_1 c_1 + 3q_1 c_2 - \frac{6c_1}{7}) A^{-1} \Gamma^{zi} - \frac{6}{7} \not\partial \Phi \Gamma^i + \frac{1}{28} \not\partial \mathcal{G}^i + \frac{5}{168} \Gamma \not\partial \mathcal{G}^i \right] \\ & \quad \left[-\frac{1}{\ell} \left(\frac{q_1 c_1 + q_1 c_2}{2} - \frac{c_1}{7} \right) A^{-1} \Gamma_{zi} - \frac{1}{7} \Gamma_i \not\partial \Phi - \frac{5}{56} \not\partial \mathcal{G}_i + \frac{1}{84} \Gamma \not\partial \mathcal{G}_i \right] \chi_+ \rangle \\ &= \langle \chi_+, \left[-\frac{3}{7\ell^2} (7q_1 c_1 + 7q_1 c_2 - 2c_1)^2 A^{-2} - \frac{12}{7} (d\Phi)^2 - \frac{1}{21} \partial_i \Phi \Gamma \not\partial \mathcal{G}^i \right. \\ & \quad \left. - \frac{1}{42\ell} (7q_1 c_2 + 7q_1 c_1 - 2c_1) A^{-1} \not\partial \mathcal{G} \Gamma_z - \frac{1}{504} \not\partial \mathcal{G} \not\partial \mathcal{G} - \frac{1}{24} G^2 \right] \chi_+ \rangle. \end{aligned} \quad (5.1.51)$$

The last term on the right hand side of (B.8.6) is the only term involving derivatives of the fields other than Φ and the second derivative of Φ . However, we can use the Bianchi identity and the Φ field equation to rewrite this term as,

$$\begin{aligned} & \langle \chi_+, \nabla_i \left[-2\Gamma^i \Gamma^j \Psi_j^{(+)} - 2A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} + 2\Gamma^i \mathcal{A} \right] \chi_+ \rangle \\ &= \langle \chi_+, [2\nabla^2 \Phi + \frac{1}{48} dG] \chi_+ \rangle \\ &= \langle \chi_+, [\frac{4}{\ell^2} A^{-2} + 4(d\Phi)^2 - \frac{1}{6} G^2] \chi_+ \rangle, \end{aligned} \quad (5.1.52)$$

and we can use the scalar part of the Einstein equation to rewrite the second term on the right hand side of (B.8.6) as

$$\frac{1}{2}R^{(7)}\|\chi_+\|^2 = \left\langle \chi_+, \left[-\frac{2}{\ell^2}A^{-2} - 2(d\Phi)^2 + \frac{5}{24}G^2 \right] \chi_+ \right\rangle. \quad (5.1.53)$$

Now we write the sum of (B.7.40), (B.7.42), and (B.7.43), as a linear combination of $\|\mathbb{B}^{(+)}\chi_+\|^2$, $\langle \Gamma_z \mathbb{B}^{(+)}\chi_+, \mathcal{A}^{(+)}\chi_+ \rangle$, and $\|\mathcal{A}^{(+)}\chi_+\|^2$. In particular, the sum of (B.7.40), (B.7.42), and (B.7.43) is given by

$$\begin{aligned} \left\langle \chi_+, \left[\frac{1}{\ell^2} \left(\frac{2}{7} - 42q_1^2 + 12q_1 + 12q_1c_1c_2 - 42q_1^2c_1c_2 \right) A^{-2} + \frac{2}{7}(d\Phi)^2 - \frac{1}{21}\partial_i\Phi\Gamma\mathcal{G}^i \right. \right. \\ \left. \left. - \frac{1}{42\ell}(7q_1c_2 + 7q_1c_1 - 2c_1)A^{-1}\mathcal{G}\Gamma_z - \frac{1}{504}\mathcal{G}\mathcal{G} \right] \chi_+ \right\rangle, \end{aligned} \quad (5.1.54)$$

whereas

$$\begin{aligned} \|\mathbb{B}^{(+)}\chi_+\|^2 &= \frac{1+c_1c_2}{2\ell^2}\|\chi_+\|^2, \\ \langle \Gamma_z \mathbb{B}^{(+)}\chi_+, \mathcal{A}^{(+)}\chi_+ \rangle &= \left\langle \chi_+, \left[-\frac{1}{2\ell^2}(1+c_1c_2)A^{-1} - \frac{1}{24\ell}(c_1+c_2)\mathcal{G}\Gamma_z \right] \chi_+ \right\rangle, \\ \|\mathcal{A}^{(+)}\chi_+\|^2 &= \left\langle \chi_+, \left[(d\Phi)^2 + \frac{1}{\ell^2}A^{-2} - \frac{1}{6}\partial_i\Phi\Gamma\mathcal{G}^i + \frac{c_1}{6\ell}A^{-1}\mathcal{G}\Gamma_z \right. \right. \\ &\quad \left. \left. - \frac{1}{144}\mathcal{G}\mathcal{G} \right] \chi_+ \right\rangle. \end{aligned} \quad (5.1.55)$$

It follows that

$$\begin{aligned} \nabla^2\|\chi_+\|^2 - 2\partial_i\Phi\nabla^i\|\chi_+\|^2 &= \left\| \hat{\nabla}^{(+,q_1,q_2)}\chi_+ \right\|^2 + 28(q_1 - 3q_1^2)A^{-2}\|\mathbb{B}^{(+)}\chi_+\|^2 \\ &+ 4q_1A^{-1}\langle \Gamma_z \mathbb{B}^{(+)}\chi_+, \mathcal{A}\chi_+ \rangle + \frac{2}{7}\|\mathcal{A}\chi_+\|^2. \end{aligned} \quad (5.1.56)$$

This expression is suitable to apply the Hopf maximum principle on the scalar function $\|\chi_+\|^2$ on M^7 as for $0 < q_1 < \frac{2}{7}$ the right hand side of this equation is positive definite. Assuming that the conditions required for the maximum principle on the fields and M^7 apply, e.g. the fields are smooth and M^7 is compact without boundary, the only solutions to the above equation are that $\|\chi_+\|^2$ is constant, and that,

$$\nabla^{(\pm)}\chi_{\pm} = 0, \quad \mathcal{A}^{(\pm)}\chi_{\pm} = 0, \quad \mathbb{B}^{(\pm)}\chi_{\pm} = 0. \quad (5.1.57)$$

Thus χ_+ is a Killing spinor which establishes the theorem.

5.2 AdS₃ backgrounds with $dH \neq 0$

We now consider first order α' corrections to the equations of heterotic supergravity, including dH . It is not a trivial matter to extend the above results to include these α' terms, however it is tractable, and we will find that supersymmetry enhancement and Lichnerowicz-type results still hold.

5.2.1 Bianchi identities, field equations and KSEs

Let us first consider the modifications that occur in the Bianchi identity, field equations and KSEs of heterotic theory up to two loops in sigma model perturbation theory⁶. The anomaly cancelation mechanism requires the modification of the Bianchi identity for H as

$$dH = -\frac{\alpha'}{4} \left[\text{tr}(\tilde{R} \wedge \tilde{R}) - \text{tr}(F \wedge F) \right] + \mathcal{O}(\alpha'^2) , \quad (5.2.1)$$

where \tilde{R} is the curvature of a connection on the spacetime M which will not be specified at this stage, F is the curvature of the gauge sector connection of the heterotic theory and α' is the string tension which also has the role of the loop parameter. Thus dH is expressed as the difference of two Pontryagin forms, one is that of the tangent space of space-time and the other is that of the gauge sector bundle. Furthermore, global anomaly cancelation requires in addition that the form on the right-hand-side of the anomalous Bianchi identity represents the trivial cohomology class in $H^4(M)$. This statement is modified upon the addition of NS5-brane sources but this will not be considered here.

In addition to the modification of the Bianchi identity, the field equations also get modified. In particular up to two loops in sigma model perturbation theory [72], the dilaton, 2-form gauge potential, and gauge sector connection field equations read

$$\begin{aligned} \nabla^2 \Phi &= 2(d\Phi)^2 - \frac{1}{12} H^2 \\ &\quad + \frac{\alpha'}{16} \left[\tilde{R}_{MNST} \tilde{R}^{MNST} - F_{MNab} F^{MNab} \right] + \mathcal{O}(\alpha'^2), \\ \nabla^R H_{MNR} &= 2\partial^R \Phi H_{MNR} + \mathcal{O}(\alpha'^2), \\ \nabla^N F_{MN} + [A^N, F_{MN}] &= 2\partial^N \Phi F_{MN} + \frac{1}{2} H_{MNQ} F^{NQ} + \mathcal{O}(\alpha') , \end{aligned} \quad (5.2.2)$$

and the Einstein equation is

$$R_{MN} = \frac{1}{4} H_{MN}^2 - 2\nabla_M \nabla_N \Phi - \frac{\alpha'}{4} \left[\tilde{R}_{MLST} \tilde{R}_N{}^{LST} - F_{MLab} F_N{}^{Lab} \right] + \mathcal{O}(\alpha'^2) . \quad (5.2.3)$$

Furthermore, the KSEs [73] are

$$\begin{aligned} \nabla_M \epsilon - \frac{1}{8} \not{H}_M \epsilon &= 0 + \mathcal{O}(\alpha'^2) , \\ (\not{\partial} \Phi - \frac{1}{12} \not{H}) \epsilon &= 0 + \mathcal{O}(\alpha'^2) , \\ \not{F} \epsilon &= 0 + \mathcal{O}(\alpha') . \end{aligned} \quad (5.2.4)$$

In particular observe that the KSEs have the same form up to two loops in sigma model perturbation theory as that at the zeroth order. It is not known how these equations are modified at higher orders. The gauge indices of F have been suppressed.

Before we proceed with the investigation of AdS_3 backgrounds, let us specify \tilde{R} . In perturbative heterotic theory, the choice of \tilde{R} is renormalization scheme dependent. In other words,

⁶We use the conventions and normalization of the field equations, Bianchi identities and KSEs of [71].

one can choose as \tilde{R} the curvature of any connection on M . However in most applications \tilde{R} is chosen to be the curvature \check{R} of the $\check{\nabla} = \nabla - \frac{1}{2}H$ connection on the spacetime. It is known that this choice has some key advantages. In particular it is required for the cancelation of world sheet supersymmetry anomaly [74] and also for the consistency of the anomalous Bianchi identity with the modified Einstein equations for supersymmetric backgrounds. This has been used in the calculations of [75, 71] and recently emphasized [76]. The property of \check{R} which is used to establish these is that \check{R} satisfies instanton-like conditions, i.e. it satisfies the same conditions, to zeroth order in α' , as those implied on F by the gaugino KSE. To see this, consider the identity

$$\hat{R}_{MN,RS} = \check{R}_{RS,MN} - \frac{1}{2}dH_{MNR S} . \quad (5.2.5)$$

The integrability condition of the gravitino KSE gives $\hat{R}_{MN,RS}\Gamma^{RS}\epsilon = 0$. As the right-hand-side of the anomalous Bianchi identity is of order α' , it follows from (5.2.5) that, to zeroth order in α' , $\check{R}_{MN,RS}\Gamma^{MN}\epsilon = 0$ or equivalently $\check{R}\epsilon = 0$ after suppressing the $SO(9,1)$ gauge indices. This is the same condition as that satisfied by the curvature of the gauge sector F in (5.2.4).

To find solutions in the perturbative case, it is understood that the fields and Killing spinors are expanded in α' schematically as

$$g = g^0 + \alpha' g^1 + \mathcal{O}(\alpha'^2) , \quad \epsilon = \epsilon^0 + \alpha' \epsilon^1 + \mathcal{O}(\alpha'^2) , \quad (5.2.6)$$

and similarly for the 3-form field strength, gauge potential and dilaton. Then the field equations and KSEs are solved order by order in α' to find the correction to the zeroth order fields.

Next consider the case that the corrections to the heterotic theory are taken to be exact up to and including two loops. In such a case, α' is not an expansion parameter. The anomalous Bianchi identity (5.2.1), field equations, (5.2.2) (5.2.3), and KSEs (5.2.4) do not receive further corrections from the ones that have been explicitly stated. However consistency of the anomalous Bianchi identity with the field equations requires that \tilde{R} satisfies the same conditions as those implied by the KSEs on the curvature F of the gauge connection, ie $\check{R}\epsilon = 0$ after suppressing the gauge indices. It is not apparent that such a connection always exists but there are existence theorems in many cases of interest. Notice also the difference from the perturbation theory as \tilde{R} cannot be identified with \check{R} . This is because dH does not vanish in the right-hand-side of (5.2.5).

5.2.2 AdS₃ backgrounds in perturbation theory

Suppose that the symmetries of AdS₃ remain symmetries of the background after the α' corrections are taken into account. In such a case, the fields up to two loops in perturbation theory will decompose as

$$\begin{aligned} ds^2 &= 2e^+e^- + A^2 dz^2 + ds^2(M^7) + \mathcal{O}(\alpha'^2) , \\ H &= AXe^+ \wedge e^- \wedge dz + G + \mathcal{O}(\alpha'^2) . \end{aligned} \quad (5.2.7)$$

This assumption is justified later. Furthermore, the field equations (5.2.2) and (5.2.3) read

$$\begin{aligned}\nabla^2\Phi &= -3A^{-1}\partial_i A\partial^i\Phi + 2(d\Phi)^2 - \frac{1}{12}G^2 + \frac{1}{2}X^2 \\ &\quad + \frac{\alpha'}{16}[\check{R}_{ij,k\ell}\check{R}^{ij,k\ell} - F_{ijab}F^{ijab}] + \mathcal{O}(\alpha'^2), \\ \nabla^k G_{ijk} &= -3A^{-1}\partial^k A G_{ijk} + 2\partial^k\Phi G_{ijk} + \mathcal{O}(\alpha'^2),\end{aligned}\tag{5.2.8}$$

and the AdS component of the Einstein equation is unchanged,

$$\nabla^2\ln A = -\frac{2}{\ell^2}A^{-2} - 3A^{-2}(dA)^2 + 2A^{-1}\partial_i A\partial^i\Phi + \frac{1}{2}X^2 + \mathcal{O}(\alpha'^2),\tag{5.2.9}$$

while component on M^7 is now,

$$\begin{aligned}R_{ij}^{(7)} &= 3\nabla_i\nabla_j\ln A + 3A^{-2}\partial_i A\partial_j A + \frac{1}{4}G_{ik_1k_2}G_j{}^{k_1k_2} - 2\nabla_i\nabla_j\Phi \\ &\quad - \frac{\alpha'}{4}[\check{R}_{ik,st}\check{R}_j{}^{k,st} - F_{ikab}F_j{}^{kab}] + \mathcal{O}(\alpha'^2),\end{aligned}\tag{5.2.10}$$

where $i, j, k, \ell = 1, 2, \dots, 7$ and we have assumed that \check{R} and F do not have components along the AdS_3 directions. As we shall see, this will follow from the KSEs.

In addition, one finds that

$$\begin{aligned}R^{(7)} &= 3\nabla^2\ln A + 3A^{-2}(dA)^2 + \frac{1}{4}G^2 - 2\nabla^2\Phi \\ &\quad - \frac{\alpha'}{4}[\check{R}_{ij,k\ell}\check{R}^{ij,k\ell} - F_{ijab}F^{ijab}] + \mathcal{O}(\alpha'^2) \\ &= -\frac{6}{\ell^2}A^{-2} - 6A^{-2}(dA)^2 + \frac{5}{12}G^2 + \frac{1}{2}X^2 + 12A^{-1}\partial_i A\partial^i\Phi - 4(d\Phi)^2 \\ &\quad - \frac{3\alpha'}{8}[\check{R}_{ij,k\ell}\check{R}^{ij,k\ell} - F_{ijab}F^{ijab}] + \mathcal{O}(\alpha'^2).\end{aligned}\tag{5.2.11}$$

Similarly, the anomalous Bianchi identity of H reads

$$dG = -\frac{\alpha'}{4}[\text{tr}(\check{R} \wedge \check{R}) - \text{tr}(F \wedge F)] + \mathcal{O}(\alpha'^2).\tag{5.2.12}$$

As we shall see imposing the requirement that spacetime supersymmetry is preserved by the higher order corrections simplifies the above equations further.

5.2.3 Geometry of M^7 for backgrounds with two supersymmetries

In the perturbative approach to the heterotic string, one of the questions that arises is whether the higher order corrections preserve the spacetime supersymmetry of the zeroth order background. In other words, whether there is a renormalization scheme which preserves the spacetime supersymmetry order by order in perturbation theory. Here we shall not investigate the existence of such a scheme. Instead, we shall derive the conditions for such a scheme to exist.

We have shown that for AdS_3 backgrounds admitting two spacetime supersymmetries at zero order in α' , M^7 has a G_2 structure compatible with a connection with skew-symmetric torsion. In particular at this order $dH = 0$, and A and X are constant and $c_1 = \frac{\ell}{2}AX = \pm 1$. The geometry of M^7 at this order is described in section 5.1.3.

The contribution in the terms proportional to α' in the field equations, Bianchi identities and KSEs comes from the fields at zeroth order in α' . These depend on \check{R} and F . At zeroth order, the spacetime factorizes into a product $AdS_3 \times M^7$. Furthermore the choice of torsion on AdS_3 is such that $\hat{\nabla}|_{AdS_3}$ and $\check{\nabla}|_{AdS_3}$ are either the left or right invariant parallelizing connection; AdS_3 is a group manifold. In either case, $\hat{R}|_{AdS_3} = \check{R}|_{AdS_3} = 0$. Therefore the contribution in the α' terms of field equations, Bianchi identities and KSEs comes only from the $\check{R}^{(7)}$ curvature of M^7 . Furthermore, the KSEs imply that the gauge curvature F does not have components along AdS_3 and is invariant under the isometries of AdS_3 up to gauge transformations. As a result all gauge invariant tensors constructed from F are tensors on M^7 which do not depend on the coordinates of AdS_3 . These justify the choice of \check{R} and F made in the previous section.

As the form of the gravitino KSE remains the same up to order α'^2 , this implies that A and X are constant up to that order and that again $c_1 = \frac{\ell}{2}AX = \pm 1$. Furthermore the metric and torsion of AdS_3 does not receive corrections at one loop, the form of the fields remains as in (5.2.7) up to order $\mathcal{O}(\alpha'^2)$. The background remains factorized as $AdS_3 \times M^7$ up to that order as well. Imposing all the above conditions on the fields, one finds that the anomalous Bianchi identity and field equations are simplified as in appendix B.

Next focusing on the geometry of M^7 , M^7 admits a G_2 structure compatible with a connection $\hat{\nabla}$ with skew-symmetric torsion G . As a consequence of the gravitino and dilatino KSEs, G is as given in (5.1.27) up to order α'^2 . Moreover all the KSEs and field equations are satisfied provided that

$$d(e^{2\Phi} \star_7 \varphi) = 0 + \mathcal{O}(\alpha'^2) , \quad dG = -\frac{\alpha'}{4} \left[\text{tr}(\check{R}^{(7)} \wedge \check{R}^{(7)}) - \text{tr}(F \wedge F) \right] + \mathcal{O}(\alpha'^2) . \quad (5.2.13)$$

The first condition is required for the existence of a connection with skew-symmetric torsion which is compatible with the G_2 structure on M^7 while the second condition arises from the anomalous Bianchi identity. We have also assumed as in the $dH = 0$ case that all solutions ϵ of the gravitino KSEs are also solutions of the gaugino KSE, $\not{F}\epsilon = 0$. In this case, this implies that F is a G_2 instanton on M^7 , and so it satisfies

$$F_{ij} = \frac{1}{2} \star_7 \varphi_{ij}{}^{km} F_{km} + \mathcal{O}(\alpha') , \quad (5.2.14)$$

where we have suppressed the gauge indices. This summarizes the geometry of M^7 up to order α'^2 .

5.2.4 Extended supersymmetry

Next let us investigate the geometry of AdS_3 backgrounds preserving 4, 6 and 8 supersymmetries up to order α'^2 . The geometry of the associated zeroth order backgrounds for which $dH = 0$ has already been described in section 5.1.4.

Four supersymmetries

These backgrounds are a special case of those we have described in the previous section that preserve two supersymmetries. As a result up to order α'^2 , the geometry is a product $AdS_3 \times M^7$. The presence of two more supersymmetries restricts further the geometry of M^7 . As the form of the gravitino and dilatino KSEs remain the same as that of the zeroth order fields, the geometric restrictions on the geometry of M^7 are similar to those in section 5.1.4. The only difference here is that $dH \neq 0$. In particular, M^7 has an $SU(3)$ structure compatible with a connection with skew-symmetric torsion. So it admits a Killing vector field ξ such that

$$i_\xi G = k^{-1}dw + O(\alpha'^2), \quad i_\xi F = 0 + O(\alpha') , \quad (5.2.15)$$

where $w(\xi) = k$. Moreover, M^7 can be locally described as a circle fibration of a conformally balanced, $\theta_\omega = 2d\Phi$, KT manifold B^6 with Hermitian form ω whose canonical bundle admits a connection $k^{-1}w$, such that

$$dw^{(2,0)} = 0 + O(\alpha'^2), \quad dw_{ij}\omega^{ij} = -2k^2 + O(\alpha'^2), \quad (5.2.16)$$

i.e. the canonical bundle is holomorphic and the connection satisfies the Hermitian-Einstein instanton condition, and in addition

$$\hat{\rho}_{(6)} = dw + O(\alpha'^2), \quad k^{-2}dw \wedge dw + dG_{(6)} = -\frac{\alpha'}{4} \left[\text{tr}(\tilde{R}^{(\tau)} \wedge \tilde{R}^{(\tau)}) - \text{tr}(F \wedge F) \right] + O(\alpha'^2), \quad (5.2.17)$$

where

$$\hat{\rho}_{ij}^{(6)} = \frac{1}{2} \hat{R}_{ij}^{(6)}{}^k{}_m I^m{}_k + O(\alpha'^2), \quad (5.2.18)$$

is the curvature of the canonical bundle induced from the connection with torsion $G_{(6)} = -i_I d\omega$ on B^6 and I is the complex structure of B^6 . The first condition is required for M^7 to admit an $SU(3)$ structure compatible with the connection with skew-symmetric torsion G and the second condition is required by the anomalous Bianchi identity (5.2.13), where now $i, j, k, m = 1, 2, \dots, 6$. It is understood that the expression in the right-hand-side of the second equation in (5.2.17) is evaluated at the zeroth order fields. The metric and torsion on M^7 are given from those of B^6 as in (5.1.33) but now of course the fields on B^6 obey the equations (5.2.17) above.

Six supersymmetries

The presence of additional supersymmetries restricts the geometry of M^7 further. In particular, the spacetime is still a product $AdS_3 \times M^7$ up to order α'^2 . The geometry of the zeroth order configuration has already been described in section 5.1.4 and so M^7 is locally a S^3 fibration over a 4-dimensional manifold B^4 . As the gravitino and dilatino KSEs have the same form up to order α'^2 as the zeroth order equations, it is expected that M^7 admits three $\hat{\nabla}$ -parallel vector

bilinears $\xi_{r'}$, $r' = 1, 2, 3$. Thus $\xi_{r'}$ are isometries of the metric on M^7 and $i_{\xi_{r'}}H = k^{-1}dw_{r'}$ up to order α'^2 , where $w_{r'}(\xi_{s'}) = k\delta_{r's'}$. As the geometry of the spacetime is a product up to α'^2 , these commute with the isometries of AdS_3 . However, the gravitino and dilatino KSEs do not determine the Lie bracket algebra of $\xi_{r'}$'s.

To determine $[\xi_{r'}, \xi_{s'}]$, first note that the commutator of two isometries is an isometry. Then using $\hat{\nabla}\xi_{r'} = 0$, we can establish the identities

$$k^{-1}w_{[\xi_{r'}, \xi_{s'}]} = i_{\xi_{r'}}i_{\xi_{s'}}H, \quad i_{[\xi_{r'}, \xi_{s'}]}H = k^{-1}dw_{[\xi_{r'}, \xi_{s'}]} + i_{\xi_{r'}}i_{\xi_{s'}}dH. \quad (5.2.19)$$

Next note that $i_{\xi_{r'}}i_{\xi_{s'}}dH = 0 + \mathcal{O}(\alpha'^2)$. This follows from the fact that both \check{R} and F contribute in dH via the zeroth order fields and so as a consequence of the gravitino and gaugino KSEs, $i_{\xi_{r'}}\check{R} = i_{\xi_{r'}}F = 0$. In fact F has to be a anti-self-dual instanton in the directions transverse to AdS_3 and $\xi_{r'}$. As a consequence, the commutator $[\xi_{r'}, \xi_{s'}]$ is $\hat{\nabla}$ -parallel up to order α'^2 . If $[\xi_{r'}, \xi_{s'}]$ is not expressed in terms of $\xi_{r'}$, the holonomy of $\hat{\nabla}$ is reduced to $\{1\}$ implying that the zeroth order backgrounds are group manifolds. Such backgrounds preserve 8 supersymmetries and will be investigated below. Thus $[\xi_{r'}, \xi_{s'}]$ must close on $\xi_{t'}$. Furthermore, one can use the Bianchi identity

$$\hat{R}_{M[N, PQ]} = -\frac{1}{3}\hat{\nabla}_M H_{NPQ} + \frac{1}{6}dH_{MNPQ}, \quad (5.2.20)$$

to show that $dw^{r'}$ restricted on the directions transverse to AdS_3 and $\xi_{r'}$ is $\hat{\nabla}$ -parallel. Then an analysis similar to that we have done for heterotic horizons [20] reveals that $\xi_{r'}$ close to a $\mathfrak{su}(2)$ algebra. As a result, M^7 is locally a S^3 fibration over a 4-dimensional manifold B^4 . The geometry can be described exactly as in the zeroth order case but the various formulae are now valid up to order α'^2 . The only modification occurs in the equation for the dilaton which now reads

$$\hat{\nabla}^2 e^{2\Phi} = -\frac{1}{2}(\mathcal{F}^{\text{ad}})^2 + \frac{3}{8}k^2 e^{4\Phi} + \frac{\alpha'}{8}(\tilde{R}^{(4)2} - F^2) + \mathcal{O}(\alpha'^2), \quad (5.2.21)$$

where the inner products are taken with respect to the $d\hat{s}^2$ metric. The additional α' contribution is due to the anomalous Bianchi identity of H .

Eight supersymmetries

The backgrounds with 8 supersymmetries can be investigated in a way similar to those with 6 supersymmetries described in the previous section. However there are some differences. As we have already mentioned at zeroth order in α' , section 5.1.4, B^4 is a hyper-Kähler manifold and $\mathcal{F}^{\text{sd}} = 0$. Up to order α'^2 , the spacetime remains a product $AdS_3 \times M^7$. The investigation of the closure properties of the three $\hat{\nabla}$ -parallel vector field $\xi_{r'}$ on M^7 is not necessary. This is because it is a consequence of the gravitino and dilatino KSEs that these vector fields close to a $\mathfrak{su}(2)$ algebra [68]. The metric and torsion are given as in (5.1.35) but now the formulae are valid up to order α'^2 . The only modification from the zeroth order equations is that the dilaton equation

now reads

$$\tilde{\nabla}^2 e^{2\Phi} = -\frac{1}{2}(\mathcal{F}^{\text{ad}})^2 + \frac{\alpha'}{8}(\tilde{R}^{(4)2} - F^2) + \mathcal{O}(\alpha'^2) , \quad (5.2.22)$$

where the metric $d\hat{s}(B^4)$ is the zeroth order hyper-Kähler metric and the inner products have been taken with respect to it.

For compact B^4 , at zeroth order $\mathcal{F}^{\text{ad}} = 0$, and in this case $M^7 = S^3 \times B^4$ up to discrete identifications. As a consequence, the worldsheet action of the string factorizes into a sum of a WZW model on S^3 and a sigma model on the hyper-Kähler manifold B^4 . The latter has (4,0) worldsheet supersymmetry and as a result is ultraviolet finite [77]. However, in the presence of an anomaly, the couplings are corrected order by order in α' as a consequence of maintaining manifest (4,0) supersymmetry in perturbation theory [75].

5.2.5 Truncation to two loops

Suppose now that the theory up to two loops is exact. In such a case, the geometry of the solutions has to be re-examined as several arguments that have been applied in previous cases have been based on the closure of H either to all orders or at the zeroth order in perturbation theory. Moreover α' has been treated as an arbitrary parameter. None of these two assumptions are valid any longer. Nevertheless, there is a simplifying assumption. This is that the backgrounds have the symmetries of AdS_3 . In particular, the fields can be written as (5.1.1). The KSEs are

$$\nabla_M \epsilon - \frac{1}{8} \not{H}_M \epsilon = 0 , \quad (\not{\partial} \Phi - \frac{1}{12} \not{H}) \epsilon = 0 , \quad \not{F} \epsilon = 0 . \quad (5.2.23)$$

We also assume that the gaugino KSE has as many Killing spinors as the gravitino KSE.

Two supersymmetries

The G_2 case is rather straightforward. As the form of the gravitino and dilatino KSEs in (5.2.23) is the same as that for $dH = 0$ backgrounds and the fields are invariant under the symmetries of AdS_3 , one finds that the gravitino KSE implies that A, X are constant and $c_1 = \frac{\ell}{2} AX = \pm 1$. As a result, the geometry locally decomposes as $\text{AdS}_3 \times M^7$. The geometry of M^7 can now be described as in the perturbative case with the only difference that now the equations are exact. In particular, M^7 admits a G_2 structure compatible with a connection with skew-symmetric torsion. This G_2 structure is further restricted by the KSEs, Bianchi identities and field equations as

$$d(e^{2\Phi} \star_7 \varphi) = 0 , \quad dG = -\frac{\alpha'}{4} \left[\text{tr}(\tilde{R}^{(\tau)} \wedge \tilde{R}^{(\tau)}) - \text{tr}(F \wedge F) \right] , \quad (5.2.24)$$

where φ is the fundamental G_2 3-form, $G = k\varphi + e^{2\Phi} \star_7 d(e^{-2\Phi} \varphi)$, and \tilde{R} and F are G_2 instantons, ie

$$\tilde{R}_{ij,pq}^{(\tau)} = \frac{1}{2} \star_7 \varphi_{ij}{}^{km} \tilde{R}_{km,pq}^{(\tau)} , \quad F_{ij} = \frac{1}{2} \star_7 \varphi_{ij}{}^{km} F_{km} . \quad (5.2.25)$$

The condition on F follows from the gaugino KSE. Observe that \tilde{R} , which is no longer a G_2 instanton because of (5.2.5) and $dH \neq 0$, has now been replaced with $\tilde{R}^{(7)}$. Moreover α' in (5.2.24) is a constant rather than a parameter.

Four supersymmetries

The geometry of these backgrounds also factorizes as $AdS_3 \times M^7$. Moreover, M^7 admits a $SU(3)$ structure compatible with a connection $\hat{\nabla}$ with skew-symmetric torsion. There are 4 vector spinor bilinears and there is a basis such that 3 of them generate an $\mathfrak{sl}(2, \mathbb{R})$ symmetry of AdS_3 . As these 4 vector bilinears are $\hat{\nabla}$ -parallel, their commutator is $[\xi_a, \xi_b] = i_{\xi_a} i_{\xi_b} H$. Since the geometry factorizes as $AdS_3 \times M^7$, it turns out that the commutator of the generators of $\mathfrak{sl}(2, \mathbb{R})$ with the fourth vector bilinear vanishes, and so the symmetry algebra of the spacetime is $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{u}(1)$.

The rest of the analysis is similar to that we have described for the perturbative case. In particular, the equations (5.2.15), (5.2.16), (5.2.17) and (5.2.18) are still valid but now exactly. The only modification is in the second equation in (5.2.17) which now reads

$$k^{-2} dw \wedge dw + dG_{(6)} = -\frac{\alpha'}{4} \left[\text{tr} \left(\tilde{R}^{(6)} \wedge \tilde{R}^{(6)} \right) - \text{tr}(F \wedge F) \right], \quad (5.2.26)$$

where $\tilde{R}^{(6)}$ is a $\mathfrak{su}(3)$ instanton on B^6 , ie $\tilde{R}^{(6)}$ is a (1,1)-form and ω -traceless. This condition is also satisfied by F because of the gaugino KSE.

Six supersymmetries

The geometry factorizes as $AdS_3 \times M^7$ and M^7 admits an $SU(2)$ structure compatible with a connection with skew-symmetric torsion $\hat{\nabla}$. The spacetime admits 6 vector Killing spinor bilinears. Three of these span an $\mathfrak{sl}(2, \mathbb{R})$ symmetry of AdS_3 , and the other three $\xi_{r'}$ are $\hat{\nabla}$ -parallel on M^7 and commute with those generating the $\mathfrak{sl}(2, \mathbb{R})$. We shall argue that for non-trivial backgrounds the commutator of these three vector field must close in the set. To see this, consider the identities in (5.2.19). As $\xi_{r'}$ are Killing, their commutator is also Killing. Furthermore, the term $i_{\xi_{r'}} i_{\xi_{s'}} dH$ in the second equation in (5.2.19) vanishes. This is because we have assumed that the connections that contribute in the anomalous Bianchi identity are those that satisfy the gaugino KSE. For all these $i_{\xi_{r'}} F = i_{\xi_{r'}} \tilde{R} = 0$. As a result, if $\xi_{r'}$ and $\xi_{s'}$ are $\hat{\nabla}$ -parallel, so is the commutator $[\xi_{r'}, \xi_{s'}]$. If the commutator does not close in the set $\xi_{r'}$, the holonomy of $\hat{\nabla}$ will reduce to $\{1\}$. As a result the curvature of $\hat{\nabla}$ vanishes. If this is the case, the contribution to the anomalous Bianchi identity must vanish as well as the connections that contribute to it have zero curvature. This is implied by our assumption that all solutions to the gravitino KSE are also solutions of the gaugino one. For such backgrounds $dH = 0$ and so the spacetime is a group manifold which preserves 8 supersymmetries. Thus for backgrounds with strictly six supersymmetries, we shall take that $[\xi_{r'}, \xi_{s'}]$ closes in the set $\xi_{r'}$. Then it can be shown using (5.2.20) that the symmetry group of the spacetime generated by the vector spinor bilinears is $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{su}(2)$.

The rest of the investigation of the geometry is similar to that we have done in the perturbative case. The only difference is that now

$$\mathring{\nabla}^2 e^{2\Phi} = -\frac{1}{2}(\mathcal{F}^{\text{ad}})^2 + \frac{3}{8}k^2 e^{4\Phi} + \frac{\alpha'}{8}(\tilde{R}^{(4)2} - F^2), \quad (5.2.27)$$

where $\tilde{R}^{(4)}$ and F are anti-self-dual instantons on B^4 and the inner products are taken with respect to the $d\hat{s}^2$ metric. B^4 is a 4-manifold with vanishing self-dual Weyl tensor and metric $d\hat{s}(B^4)$.

Eight supersymmetries

The investigation of the geometry of these backgrounds is simpler than that described in the previous section for backgrounds preserving 6 supersymmetries. First the geometry factorizes as $AdS_3 \times M^7$ and M^7 admits a connection with skew-symmetric torsion compatible with a $SU(2)$ structure. As in the previous case, M^7 admits 3 $\mathring{\nabla}$ -parallel Killing spinor bilinears $\xi_{r'}$ which commute with another three which span an $\mathfrak{sl}(2, \mathbb{R})$ symmetry of AdS_3 . Furthermore the gravitino and dilatino KSEs imply that the symmetry algebra of these backgrounds is $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{su}(2)$. The analysis of the geometry proceeds as in the perturbative case. In particular, M^7 is an S^3 fibration over a hyper-Kähler manifold B^4 with metric $d\hat{s}(B^4)$. The only difference from the perturbative case is that now

$$\mathring{\nabla}^2 e^{2\Phi} = -\frac{1}{2}(\mathcal{F}^{\text{ad}})^2 + \frac{\alpha'}{8}(\tilde{R}^{(4)2} - F^2), \quad (5.2.28)$$

where $\tilde{R}^{(4)}$ and F are anti-self-dual instantons on B^4 .

5.2.6 Lichnerowicz type Theorem on σ_+ , τ_+

The Lichnerowicz type theorem has to be re-examined in the presence of α' corrections and in the case that the theory is truncated to two loops. Again, we shall focus on M^7 , and define the modified Dirac-like operator as in (5.1.40) but now $dG \neq 0$. Furthermore, we assume the Bianchi identities and field equations of appendix B but now we shall include the α' terms, replacing the $\tilde{R}^{(7)}$ terms with \tilde{R} , and replacing F with \tilde{F} where \tilde{R} and \tilde{F} are arbitrary curvatures of TM^7 and the gauge sector bundle respectively. In particular \tilde{R} and \tilde{F} are not restricted by the KSEs. For the truncated theory at two loops, we take the equations in appendix B as exact but again with $\tilde{R}^{(7)}$ and F replaced with \tilde{R} and \tilde{F} .

The derivation of (B.7.40) is unaffected, but (B.7.42) becomes

$$\begin{aligned} & \left\langle \chi_+, \nabla_i \left[-2\Gamma^i \Gamma^j \Psi_j^{(+)} - 2A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} + 2\Gamma^i \mathcal{A}^{(+)} \right] \chi_+ \right\rangle \\ &= \left\langle \chi_+, \left[2\nabla^2 \Phi + \frac{1}{48} dG \right] \chi_+ \right\rangle \\ &= \left\langle \chi_+, \left[\frac{4}{\ell^2} A^{-2} + 4(d\Phi)^2 - \frac{1}{6} G^2 + \frac{\alpha'}{8} \left[\tilde{R}_{ij,kl} \tilde{R}^{ij,kl} - \tilde{F}_{ijab} \tilde{F}^{ijab} \right] \right. \right. \\ & \quad \left. \left. + \frac{\alpha'}{32} \left[\tilde{R}_{i_1 i_2, jk} \tilde{R}_{i_3 i_4,}{}^{jk} - \tilde{F}_{i_1 i_2 ab} \tilde{F}_{i_3 i_4}{}^{ab} \right] \Gamma^{i_1 i_2 i_3 i_4} \right] \chi_+ \right\rangle, \end{aligned} \quad (5.2.29)$$

and (B.7.43) also picks up an α' term;

$$\begin{aligned} \frac{1}{2} R^{(7)} \|\chi_+\|^2 &= \langle \chi_+, \left[-\frac{2}{\ell^2} A^{-2} - 2(d\Phi)^2 + \frac{5}{24} G^2 \right. \\ &\quad \left. - \frac{3\alpha'}{16} \left[\tilde{R}_{ij,k\ell} \tilde{R}^{ij,k\ell} - \tilde{F}_{ijab} \tilde{F}^{ijab} \right] \right] \chi_+ \rangle. \end{aligned} \quad (5.2.30)$$

On combining these expressions we obtain

$$\begin{aligned} \nabla^2 \|\chi_+\|^2 - 2\partial_i \Phi \nabla^i \|\chi_+\|^2 &= \left\| \hat{\nabla}^{(+,q_1,q_2)} \chi_+ \right\|^2 + 28(q_1 - 3q_1^2) A^{-2} \left\| \mathbb{B}^{(+)} \chi_+ \right\|^2 \\ &+ 4q_1 A^{-1} \left\langle \Gamma_z \mathbb{B}^{(+)} \chi_+, \mathcal{A} \chi_+ \right\rangle + \frac{2}{7} \|\mathcal{A} \chi_+\|^2 + \frac{\alpha'}{32} \|\tilde{F} \chi_+\|^2 \\ &- \frac{\alpha'}{32} \langle \tilde{R}_{\ell_1 \ell_2, mn} \Gamma^{\ell_1 \ell_2} \chi_+, \tilde{R}_{p_1 p_2, mn} \Gamma^{p_1 p_2} \chi_+ \rangle, \end{aligned} \quad (5.2.31)$$

where we have suppressed the gauge index contraction in the $\|\tilde{F} \chi_+\|^2$ term, and $q_2 = -\frac{1}{7}$.

We shall first consider the case of perturbation theory, and set $\tilde{R} = \tilde{R}^{(7)}$. We begin by systematically analysing the conditions imposed by (5.2.31) order by order in α' .⁷

To zeroth order in α' , one obtains (provided that $0 < q_1 < \frac{2}{7}$), the conditions

$$\nabla^{(+)} \chi_+ = 0 + \mathcal{O}(\alpha'), \quad \mathcal{A}^{(+)} \chi_+ = 0 + \mathcal{O}(\alpha'), \quad \mathbb{B}^{(+)} \chi_+ = 0 + \mathcal{O}(\alpha'). \quad (5.2.32)$$

The condition $\nabla^{(+)} \chi_+ = 0 + \mathcal{O}(\alpha')$ implies the integrability condition

$$\hat{R}_{mn, \ell_1 \ell_2}^{(7)} \Gamma^{\ell_1 \ell_2} \chi_+ = 0 + \mathcal{O}(\alpha'). \quad (5.2.33)$$

This in turn implies that

$$\check{R}_{\ell_1 \ell_2, mn}^{(7)} \Gamma^{\ell_1 \ell_2} \chi_+ = 0 + \mathcal{O}(\alpha'). \quad (5.2.34)$$

It follows that the final term in (5.2.31) is in fact at least of order α'^3 , and so can be ignored.

It remains to show that (5.2.31) implies the KSEs to linear order in α' . For this consider the perturbative expansion in the fields as in (5.2.6). One can show that if one assumes that the zeroth order KSEs are imposed, (5.2.31) does not have an α' correction apart from the gaugino term, which leads to the condition

$$\tilde{F} \chi_+ = 0 + \mathcal{O}(\alpha'). \quad (5.2.35)$$

So we cannot conclude that the KSEs, apart from the gaugino, are implied from (5.2.31) to order α' . For this some control over the α'^2 terms is required which is not available. Observe that the above theorem also implies that all solutions of the gravitino and dilatino KSEs are also solutions of the gaugino one. This is because the modified Dirac-like operator $\mathcal{D}^{(+)}$ is constructed from only the gravitino and dilatino KSEs but nevertheless the above theorem implies that the gaugino KSE is implied as well.

⁷We remark that in perturbation theory, the RHS of (5.2.31) is explicitly determined only up to first order in α' . The α'^2 terms are not known, as one would require the corresponding α'^2 corrections to the Dirac operator, as well as dG and $R^{(7)}$ and $\nabla^2 \Phi$, in order to fix the α'^2 terms.

In the truncated theory, one can again formulate a Lichnerowicz type of theorem provided that one imposes by hand the condition

$$\tilde{R}_{\ell_1 \ell_2, mn} \Gamma^{\ell_1 \ell_2} \chi_+ = 0 . \quad (5.2.36)$$

This condition (taking $0 < q_1 < \frac{2}{7}$) is sufficient to ensure that the RHS of (5.2.31) can be written as a sum of positive definite terms, which must all vanish.

5.3 A no-go theorem for AdS_n , $n \geq 4$ and $n = 2$ backgrounds

There are no AdS_n , $n \geq 4$ backgrounds in heterotic theory with or without α' corrections up to two loops in sigma model perturbation theory. This includes the case for which the theory is treated as exact up to and including two loops.

The proof of this relies on the solution of the KSEs. Suppose that the fields are invariant under the symmetries of AdS_n . Then we take a basis for the spacetime as $\{\mathbf{e}^\lambda = A \bar{\mathbf{e}}^\lambda, \mathbf{e}^i\}$ where $\bar{\mathbf{e}}^\lambda$ is a basis for AdS_n , and \mathbf{e}^i is a basis for the internal space M^{10-n} . We take H to be a 3-form on M^{10-n} . The components of H , and the conformal factor A , depend only on the co-ordinates of M^{10-n} .

To proceed, consider the gravitino KSE along the AdS_n frame directions, see also appendix E. This has no contribution from the 3-form H , and can be rewritten as

$$\bar{\nabla}_\lambda \epsilon - \frac{1}{2} \Gamma_\lambda \partial_i A \Gamma^i \epsilon = 0 . \quad (5.3.1)$$

where $\bar{\nabla}$ denotes the Levi-Civita connection on AdS_n . The integrability condition of this equation implies that

$$\left(\Gamma^\lambda \bar{R}_{\mu\lambda} + (1-n)(dA)^2 \Gamma_\mu \right) \epsilon = 0 . \quad (5.3.2)$$

where $\bar{R}_{\mu\nu}$ is the Ricci tensor of $\bar{\nabla}$. However, for AdS_n , $\bar{R}_{\mu\nu} = \kappa \bar{g}_{\mu\nu}$ where \bar{g} is the metric on AdS_n , and κ is a negative constant. The integrability condition (5.3.2) is then equivalent to

$$\left(\kappa + (1-n)(dA)^2 \right) \epsilon = 0 \quad (5.3.3)$$

which admits no solution as $\kappa < 0$ and $n \geq 4$.

The above argument clearly applies for all backgrounds with $dH = 0$, and so excludes the existence of AdS_n , $n > 3$, backgrounds for the common sector and the heterotic theory for which there is not an anomalous correction to the Bianchi identity. This result is also valid for the AdS_n , $n > 3$ solutions of the truncated theory as well. It remains to investigate the existence of AdS_n , $n > 3$, backgrounds in perturbative heterotic theory with an anomalous contribution to the Bianchi identity, $dH \neq 0$. In this case, the argument above implies that at zeroth order

in α' , there are no such solutions. Furthermore, it also excludes the existence of AdS_n , $n > 3$, solutions up and including two loops in sigma model perturbation theory that preserve all the symmetries of AdS_n . However such solutions cannot completely be excluded in higher orders as it is not known how the KSEs and field equations are corrected. There is the possibility that one can start from another background which is allowed at zeroth order which then gets corrected in perturbation theory to an AdS_n , $n > 3$ solution. Although this cannot be excluded, it may be a rather remote possibility. We conclude therefore that up to order $\mathcal{O}(\alpha'^2)$ in perturbation theory there are no AdS_n , $n > 3$, solutions to heterotic theory.

It remains to investigate the existence of AdS_2 solutions. It is a consequence of the investigation of near horizon geometries in [20] that if $dH = 0$, there are no AdS_2 solutions. This result extends up to order α'^2 in perturbation theory as it is unlikely that one can start from a different zeroth order background and correct it at one-loop approximation to an AdS_2 background-though we do not have a proof for this. The existence of AdS_2 solutions for the truncated theories will be examined elsewhere.

5.4 Summary

In this chapter, it has been proven that there are no heterotic $\text{AdS}_n \times_w M^{10-n}$ backgrounds with $n \neq 3$, at either zeroth-order or first-order in α' . For these AdS_3 backgrounds, it has additionally been proven that the warp factor, A , is constant, so that they are in fact product spaces of the form $\text{AdS}_3 \times M^7$, as a consequence of the AdS integrability condition of the Killing spinor equations.

A Lichnerowicz-type theorem for AdS_3 heterotic backgrounds has been proven by the author, both at zeroth-order and at first-order in α' , which proves that the Killing spinors of these backgrounds correspond to zero modes of a Dirac-like operator. Additionally, it has been proven that the geometry of the spaces depends on the number of supersymmetries preserved. If $N = 2$ supersymmetries are preserved, the transverse space supports a G_2 structure, if $N = 4$ it supports an $SU(3)$ structure, and if $N = 6$ or $N = 8$ it supports an $SU(2)$ structure.

Chapter 6

AdS₅ Backgrounds with 24 Supersymmetries

Among the various AdS backgrounds which have been discussed, AdS₅ backgrounds are of particular interest because of their duality to four-dimensional conformal field theories. This includes Maldacena's maximally supersymmetric AdS₅ × S⁵ background in IIB supergravity, which is dual to a similarly maximally supersymmetric four-dimensional superconformal field theory. Because of this interest, many specific AdS₅ backgrounds have been found, [24, 27, 28, 29, 78, 79, 80, 81, 82, 83, 84, 50], which satisfy simplifying assumptions. For example, many are assumed to satisfy the requirement that the Killing spinors are of the factorizable form $\epsilon = \psi \otimes \xi$, where ξ is a Killing spinor on the transverse space M^5 , and ψ satisfies

$$\nabla_\mu \psi + \lambda \gamma_\mu \psi, \quad (6.0.1)$$

for some constant λ . In light of recent interest in four dimensional $\mathcal{N} = 3$ CFTs [85, 86, 87, 88], the author was interested in investigating the backgrounds which are expected to be their gravitational duals, i.e., AdS₅ backgrounds which preserve exactly $N = 24$ supersymmetries.

In chapters 4 and 3, it has been shown that AdS₅ backgrounds in IIA and IIB supergravities preserve $N = 8k$ supersymmetries, $0 \leq k \leq 4$, k an integer. This result has also been shown for 11-dimensional supergravity backgrounds [13]. All maximally supersymmetric backgrounds have been classified [37], and it has been shown that no such backgrounds exist in either 11-dimensional supergravity or IIA supergravity, even with a non-zero Romans mass. However, there is no such classification of AdS₅ backgrounds preserving less than $N = 32$ supersymmetries.

In this chapter, the non-existence of AdS₅ backgrounds which preserve $N = 24$ supersymmetries in 11-dimensional and IIA supergravities are proven, in sections 6.1 and 6.2, respectively. Additionally, it is proven that AdS₅ backgrounds in IIB supergravity are locally isometric to the maximally supersymmetric AdS₅ × M⁵ background, in section 6.3. The only assumptions used in these proofs are that the fields are smooth, that the transverse space, M^6 or M^5 , is path

connected and compact, with no boundary, and that the fields are invariant under the $\mathfrak{so}(2,4)$ symmetry of AdS_5 .

6.1 $\text{AdS}_5 \times_w M^6$ Solutions in D=11

We begin by briefly summarizing the general structure of warp AdS_5 solutions in 11-dimensional supergravity, as determined in [13], whose conventions we shall follow throughout this section. Then we shall present the proof that there are no such solutions preserving 24 supersymmetries. The metric and 4-form are given by

$$\begin{aligned} ds^2 &= 2du(dr + rh) + A^2(dz^2 + e^{2z/\ell} \sum_{a=1}^2 (dx^a)^2) + ds^2(M^6) , \\ F &= X , \end{aligned} \tag{6.1.1}$$

where we have written the solution as a near-horizon geometry [19], with

$$h = -\frac{2}{\ell} dz - 2A^{-1} dA , \tag{6.1.2}$$

(u, r, z, x^1, x^2) are the coordinates of the AdS_5 space, A is the warp factor that is function on M^6 and X is a closed 4-form on M^6 . A and X depend only on the coordinates of M^6 , ℓ is the radius of AdS_5 .

The 11-dimensional Einstein equation implies that

$$D^k \partial_k \log A = -\frac{4}{\ell^2} A^{-2} - 5 \partial^k \log A \partial_k \log A + \frac{1}{144} X^2 , \tag{6.1.3}$$

where D is the Levi-Civita connection on M^6 . The remaining components of the Einstein and gauge field equations are listed in [13], however we shall only require (6.1.3) for the analysis of the $N = 24$ solutions. In particular, (6.1.3) implies that A is everywhere non-vanishing on M^6 , on assuming that M^6 is path-connected and all fields are smooth.

We adopt the following frame conventions; \mathbf{e}^i is an orthonormal frame for M^6 , and

$$\mathbf{e}^+ = du , \quad \mathbf{e}^- = dr + rh , \quad \mathbf{e}^z = Adz , \quad \mathbf{e}^a = Ae^{z/\ell} dx^a . \tag{6.1.4}$$

We use this frame in the investigation of KSEs below.

6.1.1 The Killing spinors

The Killing spinors of AdS_5 backgrounds are given by

$$\begin{aligned} \epsilon &= \sigma_+ - \ell^{-1} \sum_{a=1}^2 x^a \Gamma_{az} \tau_+ + e^{-\frac{z}{\ell}} \tau_+ + \sigma_- + e^{\frac{z}{\ell}} (\tau_- - \ell^{-1} \sum_{a=1}^2 x^a \Gamma_{az} \sigma_-) \\ &\quad - \ell^{-1} u A^{-1} \Gamma_{+z} \sigma_- - \ell^{-1} r A^{-1} e^{-\frac{z}{\ell}} \Gamma_{-z} \tau_+ , \end{aligned} \tag{6.1.5}$$

where we have used the light-cone projections

$$\Gamma_{\pm}\sigma_{\pm} = 0, \quad \Gamma_{\pm}\tau_{\pm} = 0, \quad (6.1.6)$$

and σ_{\pm} and τ_{\pm} are 16-component spinors that depend only on the coordinates of M^6 . We do not assume that the Killing spinors factorize as Killing spinors on AdS_5 and Killing spinors on M^6 .

The remaining independent Killing spinor equations (KSEs) are:

$$D_i^{(\pm)}\sigma_{\pm} = 0, \quad D_i^{(\pm)}\tau_{\pm} = 0, \quad (6.1.7)$$

and

$$\Xi^{(\pm)}\sigma_{\pm} = 0, \quad \Xi^{(\mp)}\tau_{\pm} = 0, \quad (6.1.8)$$

where

$$\begin{aligned} D_i^{(\pm)} &= D_i \pm \frac{1}{2}\partial_i \ln A - \frac{1}{288}\Gamma\mathcal{X}_i + \frac{1}{36}\mathcal{X}_i, \\ \Xi^{(\pm)} &= -\frac{1}{2}\Gamma_z\Gamma^i\partial_i \log A \mp \frac{1}{2\ell}A^{-1} + \frac{1}{288}\Gamma_z\mathcal{X}. \end{aligned} \quad (6.1.9)$$

In particular algebraic KSEs (6.1.8) imply that σ_+ and τ_+ cannot be linearly dependent. For our Clifford algebra conventions see also appendix A.

6.1.2 Counting the Killing Spinors

In order to count the number of supersymmetries, note that if σ_+ is a solution of the σ_+ KSEs, then so is $\Gamma_{12}\sigma_+$. Furthermore, $\tau_+ = \Gamma_z\Gamma_1\sigma_+$ and $\tau_+ = \Gamma_z\Gamma_2\sigma_+$ are solutions to the τ_+ KSEs. The spinors $\sigma_+, \Gamma_{12}\sigma_+, \Gamma_z\Gamma_1\sigma_+, \Gamma_z\Gamma_2\sigma_+$ are linearly independent. The positive chirality spinors also generate negative chirality spinors σ_-, τ_- which satisfy the appropriate KSEs. This is because if σ_+, τ_+ is a solution, then so is

$$\sigma_- = A\Gamma_- \Gamma_z \sigma_+, \quad \tau_- = A\Gamma_- \Gamma_z \tau_+, \quad (6.1.10)$$

and also conversely, if σ_-, τ_- is a solution, then so is

$$\sigma_+ = A^{-1}\Gamma_+ \Gamma_z \sigma_-, \quad \tau_+ = A^{-1}\Gamma_+ \Gamma_z \tau_-. \quad (6.1.11)$$

So for a generic $AdS_5 \times_w M^6$ solution, all of the Killing spinors are generated by the σ_+ spinors, each of which gives rise to 8 linearly independent spinors via the mechanism described here. The solutions therefore preserve $8k$ supersymmetries, where k is equal to the number of σ_+ spinors.

6.1.3 Non-existence of $N = 24$ AdS_5 solutions in D=11

To consider the AdS_5 solutions preserving 24 supersymmetries, we begin by setting

$$\Lambda = \sigma_+ + \tau_+ \quad (6.1.12)$$

and defining

$$W_i = A \langle \Lambda, \Gamma_{z12} \Gamma_i \Lambda \rangle . \quad (6.1.13)$$

Then (6.1.7) implies that

$$D_{(i} W_{j)} = 0 \quad (6.1.14)$$

so W is an isometry of M^6 . In addition, the algebraic conditions (6.1.8) imply that

$$\frac{1}{288} \langle \Lambda, \Gamma X_i \Lambda \rangle - \frac{1}{2} \| \Lambda \|^2 A^{-1} D_i A - \ell^{-1} A^{-1} \langle \tau_+, \Gamma_i \Gamma_z \sigma_+ \rangle = 0 . \quad (6.1.15)$$

Also, (6.1.7) implies that

$$D_i \| \Lambda \|^2 = - \| \Lambda \|^2 A^{-1} D_i A + \frac{1}{144} \langle \Lambda, \Gamma X_i \Lambda \rangle . \quad (6.1.16)$$

Combining (6.1.15), and (6.1.16) we have

$$D_i \| \Lambda \|^2 - 2\ell^{-1} A^{-1} \langle \tau_+, \Gamma_i \Gamma_z \sigma_+ \rangle = 0 . \quad (6.1.17)$$

In addition (6.1.7) implies that

$$D^i \left(A \langle \tau_+, \Gamma_i \Gamma_z \sigma_+ \rangle \right) = 0 . \quad (6.1.18)$$

Hence, on taking the divergence of (6.1.17), we find

$$D^i D_i \| \Lambda \|^2 + 2A^{-1} D^i A D_i \| \Lambda \|^2 = 0 . \quad (6.1.19)$$

A maximum principle argument then implies that $\| \Lambda \|^2$ is constant. Substituting these conditions back into (6.1.16), we find the condition

$$i_W H = 6 \| \Lambda \|^2 dA , \quad (6.1.20)$$

where

$$H = \star_6 X , \quad (6.1.21)$$

and \star_6 denotes the Hodge dual on M^6 .

To prove a non-existence theorem for $N = 24$ solutions, we consider spinors of the type

$$\Lambda = \sigma_+ + \tau_+ . \quad (6.1.22)$$

For a $N = 24$ solution, there are 12 linearly independent spinors of this type, because of the algebraic conditions (6.1.8). Next, consider the condition (6.1.20). This implies that

$$i_W dA = 0 , \quad (6.1.23)$$

where W is the isometry generated by Λ as defined in (6.1.13).

A straightforward modification of the reasoning used in [39], which we describe in Appendix B, implies that for $N = 24$ solutions, the vector fields dual to the 1-form bilinears W generated by the Λ spinors span the tangent space of M^6 . Then the condition $i_W dA = 0$ implies that A is constant, and furthermore, (6.1.20) implies that $i_W H = 0$, which also implies that $H = 0$, and so $X = 0$.

However, the Einstein equation (6.1.3) admits no AdS_5 solutions for which $dA = 0$ and $X = 0$, so there can be no $N = 24$ AdS_5 solutions.

We should remark that the two assumptions we have made on the fields to derive this result are essential. This is because any AdS_{d+1} background can locally be written as a warped product $ds^2(AdS_{d+1}) = dy^2 + A^2(y)ds^2(AdS_d)$ for some function A which has been determined in [89]. For $d = 2$, this has previously been established in [90]. As a result the maximally supersymmetric $AdS_7 \times S^4$ solution of 11-dimensional supergravity can be seen as a warped AdS_5 background. This appears to be a contradiction to our result. However, the transverse space M^6 in this case is non-compact and so it does not satisfy the two assumptions we have made.

6.2 $AdS_5 \times_w M^5$ solutions in (massive) IIA supergravity

As in the 11-dimensional supergravity investigated in the previous sections, there are no $N = 24$ AdS_5 backgrounds in (massive) IIA supergravity. We shall use the formalism and follow the conventions of [15] in the analysis that follows. Imposing invariance of the background under the symmetries of AdS_5 all the fluxes are magnetic, ie their components along AdS_5 vanish. In particular the most general AdS_5 background is

$$ds^2 = 2du(dr + rh) + A^2(dz^2 + e^{2z/\ell} \sum_{a=1}^2 (dx^a)^2) + ds^2(M^5),$$

$$G = G, \quad H = H, \quad F = F, \quad \Phi = \Phi, \quad S = S, \quad h = -\frac{2}{\ell}dz - 2A^{-1}dA, \quad (6.2.1)$$

where we have denoted the 10-dimensional fluxes and their components along M^5 with the same symbol, A is the warp factor, Φ is the dilaton and S is the cosmological constant dressed with the dilaton. A , S and Φ are functions of M^5 , while G , H and F are the 4-form, 3-form and a 2-form fluxes, respectively, which have support only on M^5 . The coordinates of AdS_5 are (u, r, z, x^a) and we introduce the frame $(\mathbf{e}^+, \mathbf{e}^-, \mathbf{e}^z, \mathbf{e}^a)$ as in (6.1.4).

The fields satisfy a number of field equations and Bianchi identities which can be found in [15]. Those relevant for the analysis that follows are the field equation for the dilaton and the field equation for G

$$D^2\Phi = -5A^{-1}\partial^i A \partial_i \Phi + 2(d\Phi)^2 + \frac{5}{4}S^2 + \frac{3}{8}F^2 - \frac{1}{12}H^2 + \frac{1}{96}G^2, \quad (6.2.2)$$

$$\nabla^\ell G_{ijk\ell} = -5A^{-1}\partial^\ell A G_{ijk\ell} + \partial^\ell \Phi G_{ijk\ell}, \quad (6.2.3)$$

respectively, and the Einstein equations both along AdS_5 and M^5

$$D^2 \ln A = -4\ell^{-2}A^{-2} - 5A^{-2}(dA)^2 + 2A^{-1}\partial_i A \partial^i \Phi + \frac{1}{96}G^2 + \frac{1}{4}S^2 + \frac{1}{8}F^2, \quad (6.2.4)$$

$$R_{ij}^{(5)} = 5\nabla_i \nabla_j \ln A + 5A^{-2}\partial_i A \partial_j A + \frac{1}{12}G_{ij}^2 - \frac{1}{96}G^2 \delta_{ij} - \frac{1}{4}S^2 \delta_{ij} + \frac{1}{4}H_{ij}^2 + \frac{1}{2}F_{ij}^2 - \frac{1}{8}F^2 \delta_{ij} - 2\nabla_i \nabla_j \Phi, \quad (6.2.5)$$

respectively, where D is the Levi-Civita connection of M^5 and $R_{ij}^{(5)}$ is the Ricci tensor of M^5 . The former is seen as the field equation for the warp factor A .

6.2.1 Killing spinor equations

The killing spinors of IIA AdS_5 backgrounds are given as in (6.1.5) where now σ_\pm and τ_\pm are 16-component spinors that depend only on the coordinates of M^5 . The remaining independent conditions are the gravitino KSEs

$$\nabla_i^{(\pm)} \sigma_\pm = 0, \quad \nabla_i^{(\pm)} \tau_\pm = 0, \quad (6.2.6)$$

the dilatino KSEs

$$\mathcal{A}^{(\pm)} \sigma_\pm = 0, \quad \mathcal{A}^{(\pm)} \tau_\pm = 0, \quad (6.2.7)$$

and the algebraic KSEs

$$\Xi_\pm \sigma_\pm = 0, \quad \Xi_\pm \tau_\pm = \mp \ell^{-1} \tau_\pm, \quad (6.2.8)$$

where

$$\begin{aligned} \nabla_i^{(\pm)} &= D_i + \Psi_i^{(\pm)}, \\ \mathcal{A}^{(\pm)} &= \not{\partial} \Phi + \frac{1}{12} \not{H} \Gamma_{11} + \frac{5}{4} S + \frac{3}{8} \not{F} \Gamma_{11} + \frac{1}{96} \not{G}, \\ \Xi_\pm &= \mp \frac{1}{2\ell} + \frac{1}{2} \not{\partial} A \Gamma_z - \frac{1}{8} A S \Gamma_z - \frac{1}{16} A \not{F} \Gamma_z \Gamma_{11} - \frac{1}{192} A \not{G} \Gamma_z, \end{aligned} \quad (6.2.9)$$

and where D is the spin connection on M^5 and

$$\Psi_i^{(\pm)} = \pm \frac{1}{2A} \partial_i A + \frac{1}{8} \not{H}_i \Gamma_{11} + \frac{1}{8} S \Gamma_i + \frac{1}{16} \not{F}_i \Gamma_{11} + \frac{1}{192} \not{G} \Gamma_i, \quad (6.2.10)$$

see appendix A for our Clifford algebra conventions. The counting of supersymmetries is exactly the same as in the D=11 supergravity described in the previous sections.

6.2.2 $N = 24$ AdS_5 solutions in (massive) IIA supergravity

Before we proceed with the analysis, the homogeneity conjecture¹ [39] together with the results [48, 91] on the classification of (massive) IIA backgrounds imply that both Φ and S are constant

¹Strictly speaking the homogeneity conjecture has not been proven for massive IIA supergravity but it is expected to hold.

functions over the whole spacetime which we shall assume from now on. Next let us set

$$\Lambda = \sigma_+ + \tau_+ , \quad (6.2.11)$$

and define

$$W_i = A \langle \Lambda, \Gamma_{zxy} \Gamma_i \Lambda \rangle . \quad (6.2.12)$$

Then (6.2.6) implies that

$$D_{(i} W_{j)} = 0 , \quad (6.2.13)$$

so W is an isometry of M^5 .

After some straightforward computation using the gravitino KSEs, one finds

$$D_i \parallel \Lambda \parallel^2 = -A^{-1} \partial_i A \parallel \Lambda \parallel^2 - \frac{1}{4} S \langle \Lambda, \Gamma_i \Lambda \rangle - \frac{1}{8} \langle \Lambda, \Gamma F_i \Gamma_{11} \Lambda \rangle - \frac{1}{96} \langle \Lambda, \Gamma G_i \Lambda \rangle \quad (6.2.14)$$

On the other hand (6.2.8) gives

$$(\not{\partial} A \Gamma_z - \frac{1}{4} A S \Gamma_z - \frac{1}{8} A \not{F} \Gamma_z \Gamma_{11} - \frac{1}{96} A \not{G} \Gamma_z) \Lambda = -\ell^{-1} \tau_+ + \ell^{-1} \sigma_+ . \quad (6.2.15)$$

Using this, (6.2.14) can be written as

$$D_i \parallel \Lambda \parallel^2 = 2\ell^{-1} A^{-1} \langle \tau_+, \Gamma_i \Gamma_z \sigma_+ \rangle . \quad (6.2.16)$$

Furthermore using (6.2.6), one can show that

$$D^i (A \langle \tau_+, \Gamma_i \Gamma_z \sigma_+ \rangle) = 0 . \quad (6.2.17)$$

Taking the covariant derivative of (6.2.16) and using the above equation, one finds that

$$D^i D_i \parallel \Lambda \parallel^2 + 2A^{-1} D^i A D_i \parallel \Lambda \parallel^2 = 0 . \quad (6.2.18)$$

This in turn implies after using the maximum principle that $\parallel \Lambda \parallel^2$ is constant.

Using the constancy of $\parallel \Lambda \parallel^2$, (6.2.14) and (6.2.16) imply that

$$-A^{-1} \partial_i A \parallel \Lambda \parallel^2 - \frac{1}{4} S \langle \Lambda, \Gamma_i \Lambda \rangle - \frac{1}{8} \langle \Lambda, \Gamma F_i \Gamma_{11} \Lambda \rangle - \frac{1}{96} \langle \Lambda, \Gamma G_i \Lambda \rangle = 0 , \quad (6.2.19)$$

and

$$\langle \tau_+, \Gamma_i \Gamma_z \sigma_+ \rangle = 0 . \quad (6.2.20)$$

Next taking the difference of the two identities below

$$\langle \tau_+, \Xi_+ \sigma_+ \rangle = 0 , \quad \langle \sigma_+, (\Xi_+ + \ell^{-1} \tau_+) \rangle = 0 , \quad (6.2.21)$$

and upon using (6.2.20), we find

$$\langle \tau_+, \sigma_+ \rangle = 0 , \quad (6.2.22)$$

ie τ_+ and σ_+ are orthogonal.

To continue, multiply $\Xi_+\Lambda = -\ell^{-1}\tau_+$ with Γ_{xy} , and using the fact $\Gamma_{xy}\tau_+$ is again a type τ_+ Killing spinor, and the equation above, one obtains that

$$W^i\partial_i A = 0 . \quad (6.2.23)$$

As straightforward modification of the argument used in [39] to prove the homogeneity conjecture, see also appendix B, one can show that the vector fields W span the tangent spaces of M^5 . As a result, the above equation implies that A is constant.

Next using the dilatino KSE (6.2.7) to eliminate the G -dependent term in (6.2.19) and that $A = \text{const}$, one finds

$$4S\langle\Lambda, \Gamma_i\Lambda\rangle + \langle\Lambda, \mathbb{F}_i\Gamma_{11}\Lambda\rangle + \frac{1}{3}\langle\Lambda, \mathbb{H}_i\Gamma_{11}\Lambda\rangle = 0 . \quad (6.2.24)$$

In what follows, we shall investigate the standard and massive IIA supergravities separately.

Standard IIA supergravity $S = 0$

In the case for which $S = 0$, the dilatino KSEs (6.2.7) imply that

$$\langle\Lambda, \mathbb{G}\Gamma_{11}\Lambda\rangle = 0 , \quad (6.2.25)$$

or equivalently, $W \wedge G = 0$. As the W span the tangent space of M^5 , it follows that $G = 0$. Then, using the dilatino KSE (6.2.7) to eliminate the F terms from (6.2.24), we obtain

$$\langle\Lambda, \mathbb{H}_i\Gamma_{11}\Lambda\rangle = 0 , \quad (6.2.26)$$

which implies that $W \wedge H = 0$. As the W span the tangent space of M^5 , it follows that $H = 0$ also. The dilaton field equation (6.2.3) then implies that $F = 0$ as well. However, for $S = 0$, $G = 0$, $H = 0$ and $F = 0$, the the warp factor field equation (6.2.4) becomes inconsistent, and so there are no AdS_5 solutions in standard IIA supergravity that preserve 24 supersymmetries.

Massive IIA supergravity $S \neq 0$

On writing $G = \star_5 X$, where X is a 1-form on M^5 , the condition

$$\frac{5}{4}S\langle\Lambda, \Gamma_{11}\Lambda\rangle + \frac{1}{96}\langle\Lambda, \mathbb{G}\Gamma_{11}\Lambda\rangle = 0 , \quad (6.2.27)$$

which is derived from the dilatino KSE (6.2.7), can be rewritten as

$$\frac{5}{4}S\langle\Lambda, \Gamma_{11}\Lambda\rangle - \frac{1}{4}A^{-1}i_W X = 0 . \quad (6.2.28)$$

Furthermore, the G field equation implies that $dX = 0$, and we assume² that $\mathcal{L}_W G = 0$ which implies $\mathcal{L}_W X = 0$. This condition, together with $dX = 0$, gives that $i_W X$ is constant. Then it follows from (6.2.28) that $\langle\Lambda, \Gamma_{11}\Lambda\rangle$ is also constant.

²The invariance of G under the vector fields constructed as Killing spinor bilinears has not been proven for massive IIA in complete generality but it is expected to hold.

On differentiating the condition $\langle \Lambda, \Gamma_{11} \Lambda \rangle = \text{const}$ using the gravitino KSEs, we obtain the condition

$$-\frac{1}{4}F_{ij}\langle \Lambda, \Gamma^j \Lambda \rangle + \frac{1}{24}\langle \Lambda, \Gamma_{11} \mathcal{G}_i \Lambda \rangle = 0 , \quad (6.2.29)$$

and hence

$$X^i F_{ij} \langle \Lambda, \Gamma^j \Lambda \rangle = 0 . \quad (6.2.30)$$

However, using an argument directly analogous to that used to show that the vector fields W span the tangent space of M^5 , it follows that the vectors $\langle \Lambda, \Gamma^j \Lambda \rangle \partial_j$ also span the tangent space of M^5 , see appendix B. Therefore,

$$i_X F = 0 . \quad (6.2.31)$$

Next, act on the right-hand-side of the dilatino equation (6.2.7) with $\not{X}\Gamma_{11}$ and take the inner product with Λ . On making use of $i_X F = 0$, we find the condition

$$\langle \Lambda, X_{\ell_1} H_{\ell_2 \ell_3 \ell_4} \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4} \Lambda \rangle = 0 , \quad (6.2.32)$$

and hence

$$\langle \Lambda, \Gamma_{11} \Gamma_{xyz} \Gamma_q \Lambda \rangle \epsilon^{q \ell_1 \ell_2 \ell_3 \ell_4} X_{\ell_1} H_{\ell_2 \ell_3 \ell_4} = 0 . \quad (6.2.33)$$

Again, as the vectors $\langle \Lambda, \Gamma_{11} \Gamma_{xyz} \Gamma^j \Lambda \rangle \partial_j$ span the tangent space of M^5 , this condition implies that

$$X \wedge H = 0 . \quad (6.2.34)$$

Another useful condition is to note that $\mathcal{L}_W X = 0$ implies that

$$\mathcal{L}_W (D^i X_i) = 0 , \quad (6.2.35)$$

and as the W span the tangent space of M^5 , it follows that $D^i X_i$ must be constant. However the integral of $D^i X_i$ over M^5 vanishes, and hence it follows that

$$D^i X_i = 0 , \quad (6.2.36)$$

ie X is co-closed. As it is also closed, X and so G are harmonic. This condition, together with $dX = 0$, imply that one can write

$$D^2 X^2 = 2D^i X^j D_i X_j + 2X^j (D_i D_j - D_j D_i) X^i = 2D^i X^j D_i X_j + 2X^i X^j R_{ij}^{(5)} . \quad (6.2.37)$$

On using the Einstein equation (6.2.5), together with the conditions $i_X F = 0$, $X \wedge H = 0$, we find

$$D^2 X^2 = 2D^i X^j D_i X_j + X^2 \left(-\frac{1}{48} G^2 - \frac{1}{2} S^2 - \frac{1}{4} F^2 + \frac{1}{6} H^2 \right) , \quad (6.2.38)$$

which can be written as

$$D^2 X^2 = 2D^i X^j D_i X_j + X^2(2S^2 + \frac{3}{2}F^2) , \quad (6.2.39)$$

on using the dilaton equation (6.2.3) to eliminate the G^2 term. As the right-hand-side of this expression is a sum of non-negative terms, an application of the maximum principle implies that X^2 is constant³ and

$$X^2 S^2 = 0 . \quad (6.2.40)$$

As $S \neq 0$, it follows that $X^2 = 0$, and hence $G = 0$. Then (6.2.27) implies that

$$\langle \Lambda, \Gamma_{11} \Lambda \rangle = 0 , \quad (6.2.41)$$

for all Killing spinors Λ . However, this is a contradiction.

To see this, let the 12-dimensional vector space spanned by the Killing spinors Λ be denoted by K . Then the above condition implies that

$$\langle \Lambda_1, \Gamma_{11} \Lambda_2 \rangle = 0 , \quad (6.2.42)$$

for all $\Lambda_1, \Lambda_2 \in K$. Denoting

$$\Gamma_{11} K = \{ \Gamma_{11} \Lambda : \Lambda \in K \} , \quad (6.2.43)$$

the condition (6.2.42) implies that $\Gamma_{11} K \subseteq K^\perp$, where

$$K^\perp = \{ \Psi : \langle \Psi, \Lambda \rangle = 0 \text{ for all } \Lambda \in K \} . \quad (6.2.44)$$

The dimension of space of all Majorana $Spin(9,1)$ spinors ζ satisfying the lightcone projection $\Gamma_+ \zeta = 0$ is 16. As K has dimension 12, K^\perp has dimension 4. As $\Gamma_{11} K$ is 12-dimensional it cannot be included in K^\perp as required by the assumption (6.2.41). Therefore there are no AdS_5 solutions in massive IIA supergravity which preserve 24 supersymmetries.

We would like to remark that the proof of this result is considerable simpler if M^5 is simply connected. As it has already been proven G is harmonic. On a simply connected M^5 , G vanishes. In such a case, (6.2.27) again implies (6.2.41). Then the non-existence of such AdS_5 backgrounds follows from the argument produced above that (6.2.41) cannot hold for all Killing spinors.

6.3 $AdS_5 \times_w M^5$ solutions in IIB supergravity

The active fields of $AdS_5 \times_w M^5$ IIB backgrounds as well as the relevant field and KSEs have been determined in [14]. In particular, in the the conventions [14], the metric and other form field strengths are

$$ds^2 = 2du(dr + rh) + A^2(dz^2 + e^{2z/\ell} \sum_{a=1}^2 (dx^a)^2) + ds^2(M^5) ,$$

³The condition $X^2 = \text{const}$ also follows from $\mathcal{L}_W X^2 = 0$ together with homogeneity.

$$G = H, \quad P = \xi, \quad F = Y \left(A^3 e^{\frac{2z}{\ell}} du \wedge (dr + rh) \wedge dz \wedge dx \wedge dy - d\text{vol}(M^5) \right), \quad (6.3.1)$$

where again we have written the background as a near-horizon geometry [19], with

$$h = -\frac{2}{\ell} dz - 2A^{-1} dA, \quad (6.3.2)$$

A is the warp factor which is a smooth function on M^5 , G is the complex 3-form, P encodes the (complexified) axion/dilaton gradients, F is the real self-dual 5-form and Y is a real scalar. The AdS_5 coordinates are (u, r, z, x^a) and we introduce the frame $(\mathbf{e}^+, \mathbf{e}^-, \mathbf{e}^z, \mathbf{e}^a)$ as in (6.1.4).

For the analysis that follows, we shall use the Bianchi identities

$$d(A^5 Y) = 0, \quad dH = iQ \wedge H - \xi \wedge \bar{H} \quad (6.3.3)$$

and the 10-dimensional Einstein equation along AdS_5 which gives the field equation

$$A^{-1} \nabla^2 A = 4Y^2 + \frac{1}{48} \|H\|^2 - \frac{4}{\ell^2} A^{-2} - 4A^{-2} (dA)^2, \quad (6.3.4)$$

for the warp factor A . The remaining Bianchi identities and bosonic field equations, which are not necessary for the investigation of $N = 24$ solutions, can be found in [14]. We also assume the same regularity assumptions as for the eleven dimensional solutions, and remark that (6.3.4) implies that A is nowhere vanishing on M^5 .

6.3.1 The Killing spinors

Solving the KSEs of IIB supergravity for $AdS_5 \times_w M^5$ backgrounds along AdS_5 , one finds that the Killing spinors can be written as in (6.1.5), where now σ_\pm and τ_\pm are Weyl $Spin(9, 1)$ spinors which depend only on the coordinates of M^5 that obey in addition the lightcone projections $\Gamma_\pm \sigma_\pm = \Gamma_\pm \tau_\pm = 0$.

The remaining independent KSEs are the gravitino parallel transport equations

$$D_i^{(\pm)} \sigma_\pm = 0, \quad D_i^{(\pm)} \tau_\pm = 0, \quad (6.3.5)$$

where

$$D_i^{(\pm)} = D_i \pm \frac{1}{2} \partial_i \log A - \frac{i}{2} Q_i \pm \frac{i}{2} Y \Gamma_i \Gamma_{xyz} + \left(-\frac{1}{96} \Gamma H_i + \frac{3}{32} \#_i \right) C^*, \quad (6.3.6)$$

together with the dilatino KSEs

$$\left(\frac{1}{24} \# + \# C^* \right) \sigma_\pm = 0, \quad \left(\frac{1}{24} \# + \# C^* \right) \tau_\pm = 0, \quad (6.3.7)$$

and some additional algebraic conditions which arise from the integration of the KSEs along the AdS_5 subspace

$$\Xi^{(\pm)} \sigma_\pm = 0, \quad \left(\Xi^{(\pm)} \pm \ell^{-1} \right) \tau_\pm = 0, \quad (6.3.8)$$

where

$$\Xi^{(\pm)} = \mp \frac{1}{2\ell} - \frac{1}{2} \Gamma_z \not{A} \pm \frac{i}{2} AY \Gamma_{xy} + \frac{1}{96} A \Gamma_z \not{H} C^* , \quad (6.3.9)$$

and C is the charge conjugation matrix. Again, we have not made any assumptions on the form of the Killing spinors.

The counting of the Killing spinors, and the way in which one can construct the σ_{\pm} , τ_{\pm} spinors from each other proceeds in exactly in the same way as for the $D = 11$ AdS_5 solutions. So, again, for a generic $AdS_5 \times_w M^5$ solution, all of the Killing spinors are generated by the σ_+ spinors, each of which gives rise to 8 linearly independent spinors. The solutions therefore preserve $8k$ supersymmetries, where k is equal to the number of σ_+ spinors.

6.3.2 $N = 24$ AdS_5 solutions in IIB

To proceed with the analysis first note that as a consequence of the homogeneity conjecture proven in [39] is that the solutions with 24 supersymmetries must be locally homogeneous, with

$$\xi = 0 . \quad (6.3.10)$$

Then, we set

$$\Lambda = \sigma_+ + \tau_+ , \quad (6.3.11)$$

and define

$$W_i = A \langle \Lambda, \Gamma_{zxy} \Gamma_i \Lambda \rangle . \quad (6.3.12)$$

Then (6.3.5) implies that

$$D_{(i} W_{j)} = 0 , \quad (6.3.13)$$

so W is an isometry of M^5 . Next, using (6.3.5), we find

$$D_i \parallel \Lambda \parallel^2 = - \parallel \Lambda \parallel^2 A^{-1} D_i A + \frac{1}{48} \text{Re} \langle \Lambda, \not{H}_i C^* \Lambda \rangle . \quad (6.3.14)$$

Furthermore, the algebraic condition (6.3.8) implies that

$$\frac{1}{48} \not{H} C^* \Lambda = (A^{-1} \Gamma^j D_j A - i Y \Gamma_{xyz}) \Lambda + \ell^{-1} A^{-1} \Gamma_z (\sigma_+ - \tau_+) . \quad (6.3.15)$$

On substituting this condition back into (6.3.14) we find

$$D_i \parallel \Lambda \parallel^2 = 2\ell^{-1} A^{-1} \text{Re} \langle \tau_+, \Gamma_i \Gamma_z \sigma_+ \rangle . \quad (6.3.16)$$

However, (6.3.5) also implies that

$$D^i \left(A \text{Re} \langle \tau_+, \Gamma_i \Gamma_z \sigma_+ \rangle \right) = 0 . \quad (6.3.17)$$

So combining this condition with (6.3.16), we find

$$D^i D_i \parallel \Lambda \parallel^2 + 2A^{-1} D^i A D_i \parallel \Lambda \parallel^2 = 0 . \quad (6.3.18)$$

A maximum principle argument then implies that $\parallel \Lambda \parallel^2$ is constant. Then (6.3.14) and (6.3.16) imply

$$- \parallel \Lambda \parallel^2 A^{-1} D_i A + \frac{1}{48} \text{Re} \langle \Lambda, \Gamma \not{H}_i C * \Lambda \rangle = 0 , \quad (6.3.19)$$

or, equivalently

$$\text{Re} \langle \tau_+, \Gamma_i \Gamma_z \sigma_+ \rangle = 0 . \quad (6.3.20)$$

Next, we shall show that the spinors σ_+ , τ_+ are orthogonal with respect to the inner product $\text{Re} \langle , \rangle$. To see this, note that (6.3.8) implies that

$$\langle \tau_+, \Xi^{(+)} \sigma_+ \rangle = 0, \quad \langle \sigma_+, (\Xi^{(+)} + \ell^{-1}) \tau_+ \rangle = 0 . \quad (6.3.21)$$

On expanding out, and subtracting these two identities, one finds that the real and imaginary parts of the resulting expression imply

$$\ell^{-1} \text{Re} \langle \tau_+, \sigma_+ \rangle + \text{Re} \langle \tau_+, \Gamma_z \Gamma^i D_i A \sigma_+ \rangle = 0 , \quad (6.3.22)$$

and

$$Y \text{Re} \langle \tau_+, \Gamma_{xy} \sigma_+ \rangle + \frac{1}{48} \text{Im} \langle \tau_+, \Gamma_z \not{H} C * \sigma_+ \rangle = 0 , \quad (6.3.23)$$

respectively. On substituting (6.3.20) into (6.3.22), we find that

$$\text{Re} \langle \tau_+, \sigma_+ \rangle = 0 . \quad (6.3.24)$$

For $N = 24$ solutions there are 6 linearly independent σ_+ spinors, and 6 linearly independent τ_+ spinors, hence the spinors of the type $\Lambda = \sigma_+ + \tau_+$ span a 12 dimensional vector space over \mathbb{R} , which we shall denote by K .

It is also particularly useful to note that the algebraic condition (6.3.8) implies

$$\begin{aligned} \frac{1}{2\ell} \langle \Lambda, \Gamma_{xy} (\tau_+ - \sigma_+) \rangle - \frac{1}{2} \langle \Lambda, \Gamma_{xyz} \Gamma^i D_i A \Lambda \rangle \\ - \frac{i}{2} A Y \parallel \Lambda \parallel^2 + \frac{A}{96} \langle \Lambda, \Gamma_{xyz} \not{H} C * \Lambda \rangle = 0 . \end{aligned} \quad (6.3.25)$$

On taking the real part of this expression, one finds

$$W^i D_i A = 0 , \quad (6.3.26)$$

where we have used the identity $\langle \Lambda, \Gamma_{xyz} \Gamma_{ijk} C * \Lambda \rangle = 0$.

The condition (6.3.26) implies that

$$dA = 0 . \quad (6.3.27)$$

This is because, by a straightforward adaptation of the analysis in [39], it follows that the isometries W generated by the spinors $\Lambda \in K$ span the tangent space of M^5 , see also appendix B. So A is constant, and the condition (6.3.19) implies that

$$\text{Re}\langle \Lambda, \Gamma H_i C * \Lambda \rangle = 0 . \quad (6.3.28)$$

To proceed further, take the divergence of this expression. On making use of the Bianchi identity for H given in (6.3.3), together with the KSE (6.3.5), we find the following condition:

$$\text{Re}\langle \Lambda, \left(\frac{9}{8} H_{\ell_1 \ell_2 i} \bar{H}_{\ell_3 \ell_4}^i \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4} - \frac{3}{4} H_{\ell_1 m n} \bar{H}_{\ell_2}{}^{mn} \Gamma^{\ell_1 \ell_2} + \frac{1}{4} H_{\ell_1 \ell_2 \ell_3} \bar{H}^{\ell_1 \ell_2 \ell_3} \right) \Lambda \rangle = 0 , \quad (6.3.29)$$

where \bar{H} is the complex conjugate of H . Furthermore, the algebraic condition (6.3.7) implies that

$$\text{Re}\langle \Lambda, \frac{1}{24} \bar{H} \not{H} \Lambda \rangle = 0 . \quad (6.3.30)$$

On expanding this expression out, and adding it to (6.3.29), one obtains the condition

$$\text{Re}\langle \Lambda, H_{\ell_1 \ell_2 i} \bar{H}_{\ell_3 \ell_4}^i \Gamma^{\ell_1 \ell_2 \ell_3 \ell_4} \Lambda \rangle = 0 , \quad (6.3.31)$$

or equivalently

$$W^i \epsilon_i^{\ell_1 \ell_2 \ell_3 \ell_4} H_{\ell_1 \ell_2 j} \bar{H}_{\ell_3 \ell_4}^j = 0 . \quad (6.3.32)$$

Again, as the W isometries span the tangent space of M^5 , one obtains

$$H_{[\ell_1 \ell_2] i} \bar{H}_{\ell_3 \ell_4}^i = 0 . \quad (6.3.33)$$

Furthermore, on substituting this condition back into

$$\langle C * \Lambda, \bar{H} \not{H} \Lambda \rangle = 0 , \quad (6.3.34)$$

which follows from (6.3.7), we find

$$\langle C * \Lambda, \Lambda \rangle \parallel H \parallel^2 = 0 . \quad (6.3.35)$$

So either $H = 0$, or $\langle C * \Lambda, \Lambda \rangle = 0$ for all $\Lambda \in K$. We shall prove that $\langle C * \Lambda, \Lambda \rangle = 0$ cannot be satisfied for all Λ .

Indeed, suppose that $\langle C * \Lambda, \Lambda \rangle = 0$ for all $\Lambda \in K$. We remark that $\langle C * \Lambda_1, \Lambda_2 \rangle$ is symmetric in Λ_1, Λ_2 , and so $\langle C * \Lambda, \Lambda \rangle = 0$ for all $\Lambda \in K$ implies that

$$\langle C * \Lambda_1, \Lambda_2 \rangle = 0 , \quad (6.3.36)$$

for all $\Lambda_1, \Lambda_2 \in K$. If we define

$$\bar{K} = \{ C * \Lambda : \Lambda \in K \}, \quad K^\perp = \{ \Psi : \text{Re}\langle \Psi, \Lambda \rangle = 0 \text{ for all } \Lambda \in K \} , \quad (6.3.37)$$

then the condition (6.3.36) implies that $\bar{K} \subset K^\perp$. However, this is not possible, because \bar{K} is 12 dimensional, whereas K^\perp is 4-dimensional. So, one cannot have $\langle C * \Lambda, \Lambda \rangle = 0$ for all $\Lambda \in K$.

It follows that

$$H = 0 \tag{6.3.38}$$

and hence the spinors Λ satisfy

$$D_i \Lambda = \left(\frac{i}{2} Q_i - \frac{i}{2} Y \Gamma_i \Gamma_{xyz} \right) \Lambda , \tag{6.3.39}$$

for constant Y , $Y \neq 0$, with

$$Y^2 = \frac{1}{\ell^2 A^2} , \tag{6.3.40}$$

as a consequence of (6.3.4). The integrability condition of (6.3.39) implies that

$$\left(R_{ijmn} - Y^2 (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \right) \Gamma^{mn} \Lambda = 0 , \tag{6.3.41}$$

where we have used the Bianchi identity $dQ = 0$. Then (6.3.41) gives that

$$\text{Re} \langle \Lambda, \Gamma_{xyz} \left(R_{ijmn} - Y^2 (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \right) \Gamma^n \Lambda \rangle = 0 , \tag{6.3.42}$$

or equivalently

$$W^n \left(R_{ijmn} - Y^2 (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \right) = 0 . \tag{6.3.43}$$

As the isometries W span the tangent space of M^5 , it follows that

$$R_{ijmn} = Y^2 (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) , \tag{6.3.44}$$

and hence M^5 is locally isometric to the round S^5 .

It follows that all (sufficiently regular) AdS_5 solutions with $N = 24$ supersymmetries are locally isometric to $AdS_5 \times S^5$, with constant axion and dilaton, and $G = 0$. This establishes that there are no distinct local geometries for IIB $AdS_5 \times M^5$ backgrounds that preserve strictly 24 supersymmetries.

6.4 Summary

In this chapter, the author has been proven that 10- and 11-dimensional AdS_5 backgrounds with compact transverse spaces cannot preserve exactly 24 supersymmetries. For IIA and 11-dimensional backgrounds, it is known that there are no maximally supersymmetric $AdS_5 \times_w M^{D-5}$ backgrounds, and so this proves that AdS_5 backgrounds preserve at most 16 supersymmetries. For IIB backgrounds, the author has proven that any AdS_5 background which preserves 24 supersymmetries is in fact maximally supersymmetric, and is locally isomorphic to $AdS_5 \times S^5$.

Chapter 7

Conclusion

In this dissertation, the author has integrated the AdS Killing spinor equations for all type II and heterotic $\text{AdS}_n \times_w M^{10-n}$ and $\mathbb{R}^{1,n-1} \times_w M^{10-n}$ backgrounds. As a consequence their allowed supersymmetry fractions have been determined, summarized in the tables below. Note that

Table 7.1: SUSY Fraction of AdS_n Backgrounds

AdS_n	SUSY fraction, N
AdS_2	$N = 2k$
AdS_3	$N = 2k$
AdS_4	$N = 4k$
AdS_5	$N = 8k$
AdS_6	$N = 16k$
AdS_7	$N = 16k$

Table 7.2: SUSY Fraction of $\mathbb{R}^{1,n-1}$ Backgrounds

$\mathbb{R}^{1,n-1}$	SUSY fraction, N
$\mathbb{R}^{1,1}$	$N = k$
$\mathbb{R}^{1,2}$	$N = 2k$
$\mathbb{R}^{1,3}$	$N = 4k$
$\mathbb{R}^{1,4}, \mathbb{R}^{1,5}$	$N = 8k$
$\mathbb{R}^{1,6}, \mathbb{R}^{1,7}, \mathbb{R}^{1,8}$	$N = 16k$
$\mathbb{R}^{1,9}$	$N = 32$

these are restrictions on the allowed supersymmetry fractions, and not all of these backgrounds necessarily exist. Supersymmetric heterotic AdS backgrounds, for example, must be AdS_3 . IIA and 11-dimensional AdS_n backgrounds are at most $n \leq 7$, while IIB backgrounds are at most

$n \leq 6$. Furthermore, the maximally supersymmetric backgrounds of each supergravity have been classified [37], which means that AdS backgrounds of other AdS dimensions cannot be maximally supersymmetric. There will likely be more similar theorems to find, as well as additional theorems akin to those presented in chapter 6, proving that certain non-maximal supersymmetry fractions cannot exist either.

Additionally, the author has proven a Lichnerowicz-type theorem for each type II and each heterotic $\text{AdS}_n \times_w M^{10-n}$ background, with the additional condition that the transverse spaces are compact. These theorems prove that the Killing spinors of each background are exactly the zeroes of a Dirac-like operator on the transverse space, and simultaneously prove that the ϵ_+ Killing spinors have constant length, a condition which is closely connected to the superalgebra properties of each background.

Even with these results, there are a number of questions related to this work that remain to be answered. For example, how many other backgrounds do similar Lichnerowicz-type theorems apply to? In general, while all Killing spinors of an arbitrary supergravity background are necessarily zeroes of any Dirac-like operator constructed from the KSEs, the converse is not necessarily true. However, given how many such theorems have now been proven, it seems likely that similar results could be found for a wide variety of supergravities. If necessary and sufficient conditions for such a correspondence to exist could be determined, then their broader context in the study of supergravities could be better understood.

Having solved the Killing spinor equations on the anti-de Sitter space of each of these backgrounds, another natural question is if this gives us any additional information about the superalgebras these backgrounds preserve. Of course, we know a priori that the AdS Killing vectors form an $\mathfrak{so}(2, n-1)$ algebra, but knowing the exact forms of the Killing spinors gives additional information about how they are related to Killing vectors, and how the isometries act on the supersymmetries. Together with the super-Jacobi identity, this information will allow the superalgebras of many of the backgrounds discussed in this dissertation to be completely determined. For most of these backgrounds, it will even determine some of the isometries of the transverse spaces.

Finally, the results in this dissertation, particularly those regarding allowed supersymmetry fractions, lay the groundwork for a complete classification of all AdS supergravity backgrounds. Restricting the search to those backgrounds with appropriate numbers of Killing spinors makes even stronger statements possible, such as the proof that all AdS_5 backgrounds which are more than 1/2-BPS are maximally supersymmetric, in chapter 6. More directly, what can be learned from the superalgebra information mentioned above will significantly restrict the possible geometries of the transverse spaces.

These results are also related to the identification of the geometries of the transverse spaces. Although the author has completely identified the supersymmetries preserved by these backgrounds, the geometric implications of these results have not been fully analyzed. It is possible that many different geometries will appear, and their identification is a problem for the future.

Appendix A

Form and Spinor Conventions

Our form conventions are as follows. Let ω be a k -form, then

$$\omega = \frac{1}{k!} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} , \quad (\text{A.0.1})$$

and

$$d\omega = \frac{1}{k!} \partial_{i_1} \omega_{i_2 \dots i_{k+1}} dx^{i_1} \wedge \dots \wedge dx^{i_{k+1}} , \quad (\text{A.0.2})$$

leading to

$$(d\omega)_{i_1 \dots i_{k+1}} = (k+1) \partial_{[i_1} \omega_{i_2 \dots i_{k+1}]} . \quad (\text{A.0.3})$$

Furthermore, we write

$$\omega^2 = \omega_{i_1 \dots i_k} \omega^{i_1 \dots i_k} , \quad \omega_{i_1 i_2}^2 = \omega_{i_1 j_1 \dots j_{k-1}} \omega_{i_2}^{j_1 \dots j_{k-1}} . \quad (\text{A.0.4})$$

Given a volume form $d\text{vol} = \frac{1}{n!} \epsilon_{i_1 \dots i_n} dx^{i_1} \wedge \dots \wedge dx^{i_n}$, the Hodge dual of ω is defined as

$$*\omega \wedge \chi = (\chi, \omega) d\text{vol} \quad (\text{A.0.5})$$

where

$$(\chi, \omega) = \frac{1}{k!} \chi_{i_1 \dots i_k} \omega^{i_1 \dots i_k} . \quad (\text{A.0.6})$$

So

$$*\omega_{i_1 \dots i_{n-k}} = \frac{1}{k!} \epsilon_{i_1 \dots i_{n-k}}^{j_1 \dots j_k} \omega_{j_1 \dots j_k} . \quad (\text{A.0.7})$$

It is well-known that for every form ω , one can define a Clifford algebra element ϕ given by

$$\phi = \omega_{i_1 \dots i_k} \Gamma^{i_1 \dots i_k} , \quad (\text{A.0.8})$$

where Γ^i , $i = 1, \dots, n$, are the Dirac gamma matrices. In addition we introduce the notation

$$\phi_{i_1} = \omega_{i_1 i_2 \dots i_k} \Gamma^{i_2 \dots i_k}, \quad \Gamma \phi_{i_1} = \Gamma_{i_1}^{i_2 \dots i_{k+1}} \omega_{i_2 \dots i_{k+1}}, \quad (\text{A.0.9})$$

as it is helpful in many of the expressions we have presented.

Additional spinor conventions used in this dissertation can be found in [92], particularly including the constructions of the Γ -matrices in terms of forms.

Appendix B

Notes and Computations

These are some of my notes from the work done for this dissertation, including the Clifford algebra computations involved in proving the Lichnerowicz-type theorems.

B.1 AdS Geometry

B.1.1 Metric and Frame Forms

The metric for $\text{AdS}_n \times_w M^{10-n}$ in lightcone coordinates is

$$ds^2 = 2du \left(dr - \frac{2r}{\ell} dz - 2rd \ln A \right) + A^2 dz^2 + A^2 e^{2z/\ell} \delta_{ab} dx^a dx^b + g_{ij} dx^i dx^j, \quad (\text{B.1.1})$$

from which we derive the frame forms

$$e^+ = du \quad (\text{B.1.2})$$

$$e^- = dr - \frac{2r}{\ell} dz - 2rd \ln A \quad (\text{B.1.3})$$

$$e^z = Adz \quad (\text{B.1.4})$$

$$e^a = A e^{z/\ell} dx^a \quad (\text{B.1.5})$$

defined such that

$$ds^2 = 2e^+ e^- + (e^z)^2 + \delta_{ab} e^a e^b. \quad (\text{B.1.6})$$

B.1.2 Spin Connection

The derivatives of the frame forms are

$$de^+ = 0 \quad (\text{B.1.7})$$

$$de^- = -\frac{2}{\ell} dr \wedge dz - 2dr \wedge d \ln A \quad (\text{B.1.8})$$

$$= -\frac{2}{\ell} A^{-1} e^- \wedge e^z - 2e^- \wedge d \ln A \quad (\text{B.1.9})$$

$$de^z = -\frac{1}{\ell} e^z \wedge d \ln A \quad (\text{B.1.10})$$

$$de^a = -e^a \wedge d \ln A - \frac{1}{\ell} A^{-1} e^a \wedge e^z. \quad (\text{B.1.11})$$

Solving $de^M + \Omega^M_N \wedge e^N = 0$, the non-zero components of the spin connection are

$$\Omega_{+-} = -\frac{1}{\ell} A^{-1} e^z - d \ln A \quad (\text{B.1.12})$$

$$\Omega_{\pm z} = \frac{1}{\ell} A^{-1} e^\mp \quad (\text{B.1.13})$$

$$\Omega_{za} = -\frac{1}{\ell} A^{-1} e_a \quad (\text{B.1.14})$$

$$\Omega_{\mu i} = A^{-1} \partial_i A e_\mu. \quad (\text{B.1.15})$$

B.1.3 Curvature

The Riemann curvature is defined by $\rho^{MN} = d\Omega^{MN} + \Omega^M_K \wedge \Omega^{KN}$,

$$\rho^{\mu\nu} = -A^{-2} \left(\frac{1}{\ell^2} + (dA)^2 \right) e^\mu \wedge e^\nu \quad (\text{B.1.16})$$

$$\rho^{\mu i} = -A^{-1} \nabla_j \nabla^i A e^\mu \wedge e^j = -(\nabla_j \nabla^i \ln A + \partial_j \ln A \partial^i \ln A) e^\mu \wedge e^j. \quad (\text{B.1.17})$$

The Ricci curvature is

$$R_{\mu\nu} = \rho_{\sigma\mu, \nu}^\sigma + \rho_{i\mu, \nu}^i \quad (\text{B.1.18})$$

$$= -(n-1)A^{-2} \left(\frac{1}{\ell^2} + (dA)^2 \right) \eta_{\mu\nu} - (\nabla^2 \ln A + A^{-2}(dA)^2) \eta_{\mu\nu} \quad (\text{B.1.19})$$

$$= \left(-\frac{n-1}{\ell^2} A^{-2} - \frac{n}{\ell^2} A^{-2} (dA)^2 - \nabla^2 \ln A \right) \eta_{\mu\nu} \quad (\text{B.1.20})$$

$$R_{ij} = \rho_{\sigma i, j}^\sigma + \rho_{ki, j}^k \quad (\text{B.1.21})$$

$$= R_{ij}^{(10-n)} - n \nabla_i \nabla_j \ln A - n A^{-2} \partial_i A \partial_j A \quad (\text{B.1.22})$$

B.2 Heterotic AdS₃

B.2.1 Fields, Field Equations and Bianchi Identities

Warped AdS₃ backgrounds are described the metric

$$ds^2 = 2\mathbf{e}^+\mathbf{e}^- + A^2 dz^2 + ds^2(M^7),$$

$$\mathbf{e} = du \quad \mathbf{e}^- = dr - \frac{2r}{\ell} dz - 2rd \ln A$$

where u, v , and z are the AdS₃ coordinates, ℓ is the AdS radius, and A is the warp factor. The fields of heterotic supergravity are compatible with the AdS symmetries if and only if the scalar field, Φ , and the warp factor, A , depend only on the M^7 coordinates, the two-form, F , is restricted to M^7 and has no coordinate dependence on AdS₃, and the three-form, H , is of the form

$$H = AXe^+ \wedge e^- \wedge dz + G,$$

where X is a scalar and G is a three-form restricted to M^7 , and neither have any coordinate dependence on AdS₃. With the fields expressed in this way, the field equations and Bianchi identity can be decomposed in terms of these components. The Bianchi identities for the gauge field strength and three form separate into three equations,

$$d(A^3 X) = 0$$

$$dG = 0$$

$$dF = 0,$$

while, from the field equations, we find that

$$\begin{aligned} \nabla^2 \Phi &= -3A^{-1} \partial_i A \partial^i \Phi + 2(d\Phi)^2 - \frac{1}{12} G^2 + \frac{1}{2} X^2 \\ \nabla^k G_{ijk} &= -3A^{-1} \partial^k A G_{ijk} + 2\partial^k \Phi G_{ijk}, \\ \nabla^j F_{ij} &= -3A^{-1} \partial^j A F_{ij} + 2\partial^j \Phi F_{ij} - \frac{1}{2} G_{ijk} F^{jk} \end{aligned}$$

and the Einstein equation separates into an AdS component,

$$\nabla^2 \ln A = -\frac{2}{\ell^2} A^{-2} - 3A^{-2} (dA)^2 + 2A^{-1} \partial_i A \partial^i \Phi + \frac{1}{2} X^2,$$

and a transverse component,

$$R_{ij}^{(7)} = 3\nabla_i \nabla_j \ln A + 3A^{-2} \partial_i A \partial_j A + \frac{1}{4} G_{ik_1 k_2} G_j^{k_1 k_2} - 2\nabla_i \nabla_j \Phi,$$

where ∇ is the Levi-Civita connection on M^7 and $R_{ij}^{(7)}$ is its curvature. Contracting the free indices of the transverse component of the Einstein equation, we can express the scalar curvature of M^7 as

$$\begin{aligned} R^{(7)} &= 3\nabla^2 \ln A + 3A^{-2} (dA)^2 + \frac{1}{4} G^2 - 2\nabla^2 \Phi \\ &= -\frac{6}{\ell^2} A^{-2} - 6A^{-2} (dA)^2 + \frac{5}{12} G^2 + \frac{1}{2} X^2 + 12A^{-1} \partial_i A \partial^i \Phi - 4(d\Phi)^2. \end{aligned}$$

B.2.2 Killing Spinor Equations

The heterotic gravitino Killing spinor equation is

$$\nabla_M \epsilon - \frac{1}{8} \not{H}_M \epsilon = 0.$$

Using the components of H defined above and the geometry of the warped product, we find that the gravitino equation restricted to the AdS_3 directions is equivalent to

$$\begin{aligned} 0 &= \partial_u \epsilon_{\pm} + A^{-1} \Gamma_{+z} (\ell^{-1} - \Xi_{\mp}) \epsilon_{\mp} \\ 0 &= \partial_r \epsilon_{\pm} - A^{-1} \Gamma_{-z} \Xi_{\pm} \epsilon_{\mp} \\ 0 &= \partial_z \epsilon_{\pm} - \Xi_{\pm} \epsilon_{\pm} + \frac{2r}{\ell} A^{-1} \Gamma_{-z} \Xi_{\pm} \epsilon_{\mp} \end{aligned}$$

where, for AdS_3 ,

$$\Xi_{\pm} = \mp \frac{1}{2\ell} + \frac{1}{2} \not{\partial} A \Gamma_z \mp \frac{1}{4} A X.$$

Furthermore, we can use these relations on Ξ_{\pm} ,

$$\begin{aligned} \Xi_{\pm} \Gamma_{z+} + \Gamma_{z+} \Xi_{\mp} &= 0 \\ \Xi_{\pm} \Gamma_{z-} + \Gamma_{z-} \Xi_{\mp} &= 0, \end{aligned}$$

to simplify the integrability conditions in these directions. We find that there is only one independent condition,

$$\begin{aligned} 0 &= \left(\Xi_{\pm}^2 \pm \frac{1}{\ell} \Xi_{\pm} \right) \epsilon_{\pm} \\ &= \left(-\frac{1}{4\ell^2} - \frac{1}{4} (dA)^2 \mp \frac{1}{4} A X \not{\partial} A \Gamma_z + \frac{1}{16} A^2 X^2 \right) \epsilon_{\pm}, \end{aligned}$$

which, if $\ell < \infty$, can only be satisfied if $dA = 0$ and

$$-\frac{1}{4\ell^2} + \frac{1}{16} A^2 X^2 = 0, \tag{B.2.1}$$

in which case

$$\Xi_{\pm} = \mp \frac{1 + c_1}{2\ell}$$

where $c_1 = \frac{\ell}{2} A X$ is either 1 or -1, as guaranteed by (B.2.1). We can easily integrate over z , finding

$$\epsilon_{\pm}(0, 0, z, y^i) = \sigma_{\pm}(y^i) + e^{\mp z/\ell} \tau_{\pm}(y^i),$$

where

$$\Xi_{\pm} \sigma_{\pm} = 0 \quad \Xi_{\pm} \tau_{\pm} = \mp \frac{1}{\ell} \tau_{\pm}.$$

For convenience, we introduce $\mathbb{B}^{(\pm)}$, which represents Ξ_{\pm} when it acts on σ_{\pm} and $\Xi_{\pm} \pm \frac{1}{\ell}$ when it acts on τ_{\pm} . The integrability condition is then succinctly expressed as $\mathbb{B}^{(\pm)} \chi_{\pm} = 0$, $\chi_{\pm} = \sigma_{\pm}, \tau_{\pm}$. Specifically,

$$\mathbb{B}^{(\pm)} = \mp \frac{c_1 + c_2}{2\ell},$$

where $c_2 = 1$ when $\chi_\pm = \sigma_\pm$ and $c_2 = -1$ when $\chi_\pm = \tau_\pm$.

Using, again, the components of H defined above, the gravitino Killing spinor restricted to M^7 is $\nabla^{(\pm)}\epsilon = \nabla_i\chi_\pm + \Psi_i^{(\pm)}\chi_\pm = 0$, where

$$\Psi_i^{(\pm)} = -\frac{1}{8}\mathcal{G}_i.$$

Additionally, we find that dilatino Killing spinor equation reduces to $\mathcal{A}^{(\pm)}\chi_\pm = 0$, where

$$\mathcal{A}^{(\pm)} = \not{\partial}\Phi \pm \frac{c_1}{\ell}A^{-1}\Gamma_z - \frac{1}{12}\mathcal{G}.$$

The independent Killing spinor equations are thus

$$\nabla^{(\pm)}\chi_\pm = 0, \quad \mathcal{A}^{(\pm)}\chi_\pm = 0, \quad \text{and} \quad \mathbb{B}^{(\pm)}\chi_\pm = 0.$$

B.2.3 Lichnerowicz-type Theorem on σ_+ , τ_+

We begin by introducing a modified version of the gravitino equation operator,

$$\hat{\nabla}_i^{(+,q_1,q_2)} = \nabla_i^{(+)} + q_1 A^{-1}\Gamma_{zi}\mathbb{B}^{(+)} + q_2 \Gamma_i \mathcal{A}^{(+)},$$

with the intention to demonstrate that, for an appropriately chosen value of q_1 and q_2 , if $\Gamma^i \hat{\nabla}_i \chi_+ = 0$, then χ_+ satisfies the gravitino and dilatino Killing spinor equations. For convenience, we also introduce an operator representing a general linear combination of the algebraic conditions,

$$\mathbb{A}^{(+,q_1,q_2)} = -q_1 A^{-1}\Gamma_z \mathbb{B}^{(+)} + q_2 \mathcal{A}^{(+)},$$

so that $\hat{\nabla}_i^{(+,q_1,q_2)} = \nabla_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2)}$, and the modified Dirac-like condition is

$$\Gamma^i \hat{\nabla}_i^{(+,q_1,q_2)} \chi_+ = \left(\Gamma^i \nabla_i + \Gamma^i \Psi_i^{(+)} + 7\mathbb{A}^{(+,q_1,q_2)} \right) \chi_+ = 0.$$

We expect to find that an equation of the form $\nabla^2 \|\chi_+\|^2 = Q(\chi_+, \chi_+)$, where the right-hand side, $Q(\chi_+, \chi_+)$ is a positive-definite quadratic function in χ_\pm . With this in mind, we now expand the Laplacian, $\nabla^2 \|\chi_+\|^2$, into two terms,

$$\nabla^2 \|\chi_+\|^2 = 2\|\nabla\chi_+\|^2 + 2\langle\chi_+, \nabla^2\chi_+\rangle.$$

The first term can be further rewritten in terms of the differential operator $\hat{\nabla}$ by completing the square,

$$\begin{aligned} 2\|\nabla\chi_+\|^2 &= 2\left\|\hat{\nabla}^{(+,q_1,q_2)}\chi_+\right\|^2 - 4\left\langle\chi_+, \left(\Psi^{(+)\dagger} + \mathbb{A}^{(+,q_1,q_2)\dagger}\Gamma^i\right)\nabla_i\chi_+\right\rangle \\ &\quad - 2\left\langle\chi_+, \left(\Psi^{(+)\dagger} + \mathbb{A}^{(+,q_1,q_2)\dagger}\Gamma^i\right)\left(\Psi_i^{(+)} + \Gamma_i\mathbb{A}^{(+,q_1,q_2)}\right)\chi_+\right\rangle \\ &= 2\left\|\hat{\nabla}^{(+,q_1,q_2)}\chi_+\right\|^2 - 4\left\langle\chi_+, \Psi^{(+)\dagger}\nabla_i\chi_+\right\rangle \\ &\quad - 2\left\langle\chi_+, \left(\Psi^{(+)\dagger} - \mathbb{A}^{(+,q_1,q_2)\dagger}\Gamma^i\right)\left(\Psi_i^{(+)} + \Gamma_i\mathbb{A}^{(+,q_1,q_2)}\right)\chi_+\right\rangle, \end{aligned}$$

while the second term can be rewritten using the property $\Gamma^i \nabla_i \Gamma^j \nabla_j \psi = \nabla^2 \psi + \frac{1}{2} R^{(7)} \psi$, and the Dirac-like condition,

$$\begin{aligned} 2\langle \chi_+, \nabla^2 \chi_+ \rangle &= 2\langle \chi_+, \Gamma^i \nabla_i (\Gamma^j \nabla_j \chi_+) \rangle + \frac{1}{2} R^{(7)} \|\chi_+\|^2 \\ &= \frac{1}{2} R^{(7)} \|\chi_+\|^2 - 2\langle \chi_+, \nabla_i (\Gamma^i \Gamma^j \Psi_j^{(+)} + 7\Gamma^i \mathbb{A}^{(+,q_1,q_2)}) \chi_+ \rangle \\ &\quad - 2\langle \chi_+, (\Gamma^i \Gamma^j \Psi_j^{(+)} + 7\Gamma^i \mathbb{A}^{(+,q_1,q_2)}) \nabla_i \chi_+ \rangle. \end{aligned}$$

Combining these, $\nabla^2 \|\chi_+\|^2$ can be rewritten as,

$$\begin{aligned} \nabla^2 \|\chi_+\|^2 &= 2\left\| \hat{\nabla}^{(+,q_1,q_2)} \chi_+ \right\|^2 + \frac{1}{2} R^{(7)} \|\chi_+\|^2 \\ &\quad + \langle \chi_+, [-4\Psi^{(+)\dagger} - 2\Gamma^i \Gamma^j \Psi_j^{(+)} - 14q_1 A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} - 14q_2 \Gamma^i \mathcal{A}^{(+)}] \nabla_i \chi_+ \rangle \\ &\quad + \langle \chi_+, -2(\Psi^{(+)\dagger} - \mathbb{A}^{(+,q_1,q_2)\dagger}) (\Psi_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2)}) \chi_+ \rangle \\ &\quad + \langle \chi_+, \nabla_i [-2\Gamma^i \Gamma^j \Psi_j^{(+)} - 14q_1 A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} - 14q_2 \Gamma^i \mathcal{A}^{(+)}] \chi_+ \rangle \end{aligned} \quad (\text{B.2.2})$$

where

$$\begin{aligned} \Psi_i^{(+)\dagger} &= \frac{1}{8} \mathcal{G}_i \\ \mathbb{B}^{(+)\dagger} &= -\frac{c_1 + c_2}{2\ell} \\ \mathcal{A}^{(+)\dagger} &= \not\partial \Phi + \frac{c_1}{\ell} A^{-1} \Gamma_z + \frac{1}{12} \mathcal{G}. \end{aligned}$$

Of the terms on the right hand side of (B.8.6), the first term is proportional to the gravitino equation squared, and so we expect that the remaining terms will be equal to some combination of the algebraic KSEs. The third term includes a derivative of χ_+ , however, and so we will attempt to write it in the form

$$\alpha^i \nabla_i \|\chi_+\|^2 + \langle \chi_+, \mathcal{F} \Gamma^i \nabla_i \chi_+ \rangle = \alpha^i \nabla_i \|\chi_+\|^2 - \langle \chi_+, \mathcal{F} (\Gamma^i \Psi_i^{(+)} + 7\mathbb{A}^{(+,q_1,q_2)}) \chi_+ \rangle.$$

In terms of the fields, the third term can be rewritten as

$$\begin{aligned} &\langle \chi_+, [-4\Psi^{(+)\dagger} - 2\Gamma^i \Gamma^j \Psi_j^{(+)} - 14q_1 A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} - 14q_2 \Gamma^i \mathcal{A}^{(+)}] \nabla_i \chi_+ \rangle \\ &= \langle \chi_+, \left[\frac{7}{\ell} A^{-1} \Gamma^{zi} (q_1 c_1 + q_1 c_2 + 2q_2 c_1) - 14q_2 \Gamma^i \not\partial \Phi \right. \\ &\quad \left. + \frac{1+14q_2}{4} \mathcal{G}^i + \frac{3+14q_2}{12} \Gamma \mathcal{G}^i \right] \nabla_i \chi_+ \rangle. \end{aligned}$$

Thus, we find that it can be separated as outlined above if and only if $q_2 = -\frac{1}{7}$. We will use this value of q_2 from here on. We therefore find that

$$\begin{aligned} &\langle \chi_+, [-4\Psi^{(+)\dagger} - 2\Gamma^i \Gamma^j \Psi_j^{(+)} - 14q_1 A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} + 2\Gamma^i \mathcal{A}] \nabla_i \chi_+ \rangle \\ &= \langle \chi_+, \left[\frac{1}{\ell} A^{-1} \Gamma^{zi} (7q_1 c_1 + 7q_1 c_2 - 2c_1) + 2\Gamma^i \not\partial \Phi \right. \\ &\quad \left. - \frac{1}{4} \mathcal{G}^i + \frac{1}{12} \Gamma \mathcal{G}^i \right] \nabla_i \chi_+ \rangle, \end{aligned}$$

and so, factoring out a Γ^i on the right,

$$\mathcal{F} = \frac{1}{\ell} A^{-1} \Gamma_z (7q_1 c_1 + 7q_1 c_2 - 2c_1) - 2\partial\Phi - \frac{1}{12} \mathcal{G}$$

and $\alpha_i = 2\partial_i \Phi$.

Now that the third term of (B.8.6) has been expressed as a term quadratic in the fields, it can readily be combined with the fourth term of (B.8.6),

$$\begin{aligned} & \left\langle \chi_+, -2 \left(\Psi^{(+)\dagger} + q_1 A^{-1} \mathbb{B}^{(+)\dagger} \Gamma^{zi} + \frac{1}{7} \mathcal{A}^{(+)\dagger} \Gamma^i + \frac{1}{2} \mathcal{F} \Gamma^i \right) \right. \\ & \quad \left. \times \left(\Psi_i^{(+)} + q_1 A^{-1} \Gamma_{zi} \mathbb{B}^{(+)} - \frac{1}{7} \Gamma_i \mathcal{A}^{(+)} \right) \chi_+ \right\rangle \\ &= \left\langle \chi_+, -2 \left[\frac{1}{\ell} \left(3q_1 c_1 + 3q_1 c_2 - \frac{6c_1}{7} \right) A^{-1} \Gamma^{zi} - \frac{6}{7} \partial\Phi \Gamma^i + \frac{1}{28} \mathcal{G}^i + \frac{5}{168} \Gamma \mathcal{G}^i \right] \right. \\ & \quad \left. \times \left[-\frac{1}{\ell} \left(\frac{q_1 c_1 + q_1 c_2}{2} - \frac{c_1}{7} \right) A^{-1} \Gamma_{zi} - \frac{1}{7} \Gamma_i \partial\Phi - \frac{5}{56} \mathcal{G}_i + \frac{1}{84} \Gamma \mathcal{G}_i \right] \chi_+ \right\rangle \\ &= \left\langle \chi_+, \left[-\frac{3}{7\ell^2} (7q_1 c_1 + 7q_1 c_2 - 2c_1)^2 A^{-2} - \frac{12}{7} (d\Phi)^2 - \frac{1}{21} \partial_i \Phi \Gamma \mathcal{G}^i \right. \right. \\ & \quad \left. \left. - \frac{1}{42\ell} (7q_1 c_2 + 7q_1 c_1 - 2c_1) A^{-1} \mathcal{G} \Gamma_z - \frac{1}{504} \mathcal{G} \mathcal{G} - \frac{1}{24} G^2 \right] \chi_+ \right\rangle. \end{aligned} \quad (\text{B.2.3})$$

The last term on the right hand side of (B.8.6) is the only term involving derivatives of the fields other than Φ and the second derivative of Φ . However, we can use the Bianchi identity and the Φ field equation to rewrite this term,

$$\begin{aligned} & \left\langle \chi_+, \nabla_i \left[-2\Gamma^i \Gamma^j \Psi_j^{(+)} - 2A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} + 2\Gamma^i \mathcal{A} \right] \chi_+ \right\rangle \\ &= \left\langle \chi_+, \left[2\nabla^2 \Phi + \frac{1}{48} d\mathcal{G} \right] \chi_+ \right\rangle \\ &= \left\langle \chi_+, \left[\frac{4}{\ell^2} A^{-2} + 4(d\Phi)^2 - \frac{1}{6} G^2 \right] \chi_+ \right\rangle, \end{aligned} \quad (\text{B.2.4})$$

and we can use the scalar part of the Einstein equation to rewrite the second term on the right hand side of (B.8.6)

$$\frac{1}{2} R^{(7)} \|\chi_+\|^2 = \left\langle \chi_+, \left[-\frac{2}{\ell^2} A^{-2} - 2(d\Phi)^2 + \frac{5}{24} G^2 \right] \chi_+ \right\rangle. \quad (\text{B.2.5})$$

Now that we've expressed the all but the first term on the right hand side of (??) as terms quadratic in the fields, we expect their sum, i.e., the sum of (B.7.40), (B.7.42), and (B.7.43), to be a linear combination of $\|\mathbb{B}^{(+)} \chi_+\|^2$, $\langle \Gamma_z \mathbb{B}^{(+)} \chi_+, \mathcal{A}^{(+)} \chi_+ \rangle$, and $\|\mathcal{A}^{(+)} \chi_+\|^2$. Indeed, comparing this sum,

$$\begin{aligned} & \left\langle \chi_+, \left[\frac{1}{\ell^2} \left(\frac{2}{7} - 42q_1^2 + 12q_1 + 12q_1 c_1 c_2 - 42q_1^2 c_1 c_2 \right) A^{-2} + \frac{2}{7} (d\Phi)^2 - \frac{1}{21} \partial_i \Phi \Gamma \mathcal{G}^i \right. \right. \\ & \quad \left. \left. - \frac{1}{42\ell} (7q_1 c_2 + 7q_1 c_1 - 2c_1) A^{-1} \mathcal{G} \Gamma_z - \frac{1}{504} \mathcal{G} \mathcal{G} \right] \chi_+ \right\rangle, \end{aligned}$$

to

$$\begin{aligned}
\left\| \mathbb{B}^{(+)} \chi_+ \right\|^2 &= \frac{1 + c_1 c_2}{2\ell^2} \|\chi_+\|^2 \\
\left\langle \Gamma_z \mathbb{B}^{(+)} \chi_+, \mathcal{A}^{(+)} \chi_+ \right\rangle &= \left\langle \chi_+, \left[-\frac{1}{2\ell^2} (1 + c_1 c_2) A^{-1} - \frac{1}{24\ell} (c_1 + c_2) \not{G} \Gamma_z \right] \chi_+ \right\rangle \\
\left\| \mathcal{A}^{(+)} \chi_+ \right\|^2 &= \left\langle \chi_+, \left[(d\Phi)^2 + \frac{1}{\ell^2} A^{-2} - \frac{1}{6} \partial_i \Phi \Gamma \not{G}^i + \frac{c_1}{6\ell} A^{-1} \not{G} \Gamma_z \right. \right. \\
&\quad \left. \left. - \frac{1}{144} \not{G} \not{G} \right] \chi_+ \right\rangle
\end{aligned}$$

we find that

$$\begin{aligned}
&\nabla^2 \|\chi_+\|^2 - 2\partial_i \Phi \nabla^i \|\chi_+\|^2 \\
&= \left\| \hat{\nabla} \chi_+ \right\|^2 + 28(q_1 - 3q_1^2) A^{-2} \left\| \mathbb{B}^{(+)} \chi_+ \right\|^2 + 4q_1 A^{-1} \left\langle \Gamma_z \mathbb{B}^{(+)} \chi_+, \mathcal{A} \chi_+ \right\rangle + \frac{2}{7} \|\mathcal{A} \chi_+\|^2.
\end{aligned}$$

The right side of this equation is positive definite if $0 < q_1 < \frac{2}{7}$. In those cases, the Hopf maximum principle tells us that $\|\chi_+\|^2$ is constant, and that,

$$\nabla^{(\pm)} \chi_{\pm} = 0, \quad \mathcal{A}^{(\pm)} \chi_{\pm} = 0, \quad \text{and} \quad \mathbb{B}^{(\pm)} \chi_{\pm} = 0,$$

i.e., χ_+ is Killing.

B.2.4 α' First Order Corrections

Most of the above analysis is entirely unaltered when we consider terms of first order in α' . The primary differences are the Bianchi identity,

$$dH = -\frac{\alpha'}{4} [\text{tr}(\check{R} \wedge \check{R}) - \text{tr}(F \wedge F)],$$

the $\nabla^2 \Phi$ field equation,

$$\nabla^2 \Phi = 2(d\Phi)^2 - \frac{1}{12} H^2 + \frac{\alpha'}{16} [\check{R}_{MNST} \check{R}^{MNST} - F_{MNab} F^{MNab}],$$

and the Einstein equation,

$$R_{MN} = \frac{1}{4} H_{MN}^2 - 2\nabla_M \nabla_N \Phi - \frac{\alpha'}{4} [\check{R}_{MRST} \check{R}_N{}^{RST} - F_{MRST} F_N{}^{RST}].$$

In these equations, \check{R} is the curvature of the connection $\nabla - \frac{1}{2} H$ and F is the two-form gauge field. Using the components of H defined above, without any assumptions about their dependence on α' , the Bianchi identity separates into two equations,

$$\begin{aligned}
d(A^3 X) &= 0 \\
dG &= -\frac{\alpha'}{4} [\text{tr}(\check{R}^{(\tau)} \wedge \check{R}^{(\tau)}) - \text{tr}(F \wedge F)].
\end{aligned}$$

The field equations can also be expressed in terms of these components,

$$\begin{aligned}\nabla^2\Phi &= -3A^{-1}\partial_i A\partial^i\Phi + 2(d\Phi)^2 - \frac{1}{12}G^2 + \frac{1}{2}X^2 \\ &\quad + \frac{\alpha'}{16}\left[\check{R}_{ij,k\ell}^{(7)}\check{R}^{(7)ij,k\ell} - F_{ijab}F^{ijab}\right] \\ \nabla^k G_{ijk} &= -3A^{-1}\partial^k A G_{ijk} + 2\partial^k\Phi G_{ijk},\end{aligned}$$

and the AdS component of the Einstein equation is unchanged,

$$\nabla^2 \ln A = -\frac{2}{\ell^2}A^{-2} - 3A^{-2}(dA)^2 + 2A^{-1}\partial_i A\partial^i\Phi + \frac{1}{2}X^2,$$

while the transverse component is now,

$$\begin{aligned}R_{ij}^{(7)} &= 3\nabla_i\nabla_j \ln A + 3A^{-2}\partial_i A\partial_j A + \frac{1}{4}G_{ik_1k_2}G_j{}^{k_1k_2} - 2\nabla_i\nabla_j\Phi \\ &\quad - \frac{\alpha'}{4}\left[\check{R}_{ik,sl}^{(7)}\check{R}_j{}^{(7)k,sl} - F_{ikab}F_j{}^{kab}\right],\end{aligned}$$

from which we find that

$$\begin{aligned}R^{(7)} &= 3\nabla^2 \ln A + 3A^{-2}(dA)^2 + \frac{1}{4}G^2 - 2\nabla^2\Phi - \frac{\alpha'}{4}\left[\check{R}_{ij,k\ell}^{(7)}\check{R}^{(7)ij,k\ell} - F_{ijab}F^{ijab}\right] \\ &= -\frac{6}{\ell^2}A^{-2} - 6A^{-2}(dA)^2 + \frac{5}{12}G^2 + \frac{1}{2}X^2 + 12A^{-1}\partial_i A\partial^i\Phi - 4(d\Phi)^2 \\ &\quad - \frac{3\alpha'}{8}\left[\check{R}_{ij,k\ell}^{(7)}\check{R}^{(7)ij,k\ell} - F_{ijab}F^{ijab}\right].\end{aligned}$$

Aside from the corrections to the fields, there are no other first order corrections to the Killing spinor equations. The derivation of (B.7.40) is therefore unaffected, but (B.7.42) becomes

$$\begin{aligned}&\left\langle\chi_+, \nabla_i\left[-2\Gamma^i\Gamma^j\Psi_j^{(+)} - 2A^{-1}\Gamma^{zi}\mathbb{B}^{(+)} + 2\Gamma^i\mathcal{A}^{(+)}\right]\chi_+\right\rangle \\ &= \left\langle\chi_+, \left[2\nabla^2\Phi + \frac{1}{48}dG\right]\chi_+\right\rangle \\ &= \left\langle\chi_+, \left[\frac{4}{\ell^2}A^{-2} + 4(d\Phi)^2 - \frac{1}{6}G^2 + \frac{\alpha'}{8}\left[\check{R}_{ij,k\ell}^{(7)}\check{R}^{(7)ij,k\ell} - F_{ijab}F^{ijab}\right]\right.\right. \\ &\quad \left.\left. + \frac{\alpha'}{32}\left[\check{R}_{i_1i_2,jk}^{(7)}\check{R}_{i_3i_4}{}^{jk} - F_{i_1i_2ab}F_{i_3i_4}{}^{ab}\right]\Gamma^{i_1i_2i_3i_4}\right]\chi_+\right\rangle,\end{aligned}$$

and (B.7.43) becomes

$$\begin{aligned}\frac{1}{2}R^{(7)}\|\chi_+\|^2 &= \left\langle\chi_+, \left[-\frac{2}{\ell^2}A^{-2} - 2(d\Phi)^2 + \frac{5}{24}G^2\right.\right. \\ &\quad \left.\left.- \frac{3\alpha'}{16}\left[\check{R}_{ij,k\ell}^{(7)}\check{R}^{(7)ij,k\ell} - F_{ijab}F^{ijab}\right]\right]\chi_+\right\rangle.\end{aligned}$$

Thus, the same theorem hold if the condition

$$\begin{aligned}&\left\langle\chi_+, \left[-\frac{\alpha'}{16}\left[\check{R}_{ij,k\ell}^{(7)}\check{R}^{(7)ij,k\ell} - F_{ijab}F^{ijab}\right]\right.\right. \\ &\quad \left.\left.+ \frac{\alpha'}{32}\left[\check{R}_{i_1i_2,jk}^{(7)}\check{R}_{i_3i_4}{}^{jk} - F_{i_1i_2ab}F_{i_3i_4}{}^{ab}\right]\Gamma^{i_1i_2i_3i_4}\right]\chi_+\right\rangle = 0\end{aligned}$$

is satisfied.

B.3 Heterotic AdS_n, $n \geq 4$

B.3.1 Field equations and Bianchi Identities

For AdS_n, $n \geq 4$, all fields are purely magnetic. The Bianchi identity is

$$dH = 0, \quad (\text{B.3.1})$$

the field equations are

$$\nabla^2 \Phi = -nA^{-1} \partial^i A \partial_i \Phi + 2(d\Phi)^2 - \frac{1}{12} H^2 \quad (\text{B.3.2})$$

$$\nabla^k H_{ijk} = -nA^{-1} \partial^k A H_{ijk} + 2\partial^k \Phi H_{ijk}, \quad (\text{B.3.3})$$

and the Einstein equation separates into an AdS component,

$$\nabla^2 \ln A = -\frac{n-1}{\ell^2} A^{-2} - nA^{-2} (dA)^2 + 2A^{-1} \partial_i A \partial^i \Phi, \quad (\text{B.3.4})$$

and a transverse component,

$$R_{ij}^{(10-n)} = n\nabla_i \nabla_j \ln A + nA^{-2} \partial_i A \partial_j A + \frac{1}{4} H_{ik_1 k_2} H_j{}^{k_1 k_2} - 2\nabla_i \nabla_j \Phi, \quad (\text{B.3.5})$$

which contracts to

$$R^{(10-n)} = n\nabla^2 \ln A + nA^{-2} (dA)^2 + \frac{1}{4} H^2 - 2\nabla^2 \Phi \quad (\text{B.3.6})$$

$$= -\frac{n(n-1)}{\ell^2} A^{-2} - n(n-1) A^{-2} (dA)^2 + \frac{5}{12} H^2 + 4nA^{-1} \partial_i A \partial^i \Phi - 4(d\Phi)^2. \quad (\text{B.3.7})$$

B.3.2 Killing Spinor Equations

The AdS-direction parallel transport equations are

$$0 = \partial_u \epsilon_{\pm} + A^{-1} \Gamma_{+z} (\ell^{-1} - \Xi_{-}) \epsilon_{\mp} \quad (\text{B.3.8})$$

$$0 = \partial_r \epsilon_{\pm} - A^{-1} \Gamma_{-z} \Xi_{+} \epsilon_{\mp} \quad (\text{B.3.9})$$

$$0 = \partial_z \epsilon_{\pm} - \Xi_{\pm} \epsilon_{\pm} + 2rA^{-1} \Gamma_{-z} \Xi_{+} \epsilon_{\mp} \quad (\text{B.3.10})$$

$$0 = \partial_a \epsilon_{+} + A^{-1} \Gamma_{za} \Xi_{+} \epsilon_{+} \quad (\text{B.3.11})$$

$$0 = \partial_a \epsilon_{-} + A^{-1} \Gamma_{za} (\Xi_{-} - \ell^{-1}) \epsilon_{-} \quad (\text{B.3.12})$$

where, for AdS_k, $k \geq 5$,

$$\Xi_{\pm} = \mp \frac{1}{2\ell} + \frac{1}{2} \not\partial A \Gamma_z. \quad (\text{B.3.13})$$

Because

$$\Xi_{\pm} \Gamma_{z+} + \Gamma_{z+} \Xi_{\mp} = 0 \quad (\text{B.3.14})$$

$$\Xi_{\pm} \Gamma_{z-} + \Gamma_{z-} \Xi_{\mp} = 0 \quad (\text{B.3.15})$$

$$\Xi_{\pm} \Gamma_{za} + \Gamma_{za} \Xi_{\pm} = \mp \ell^{-1} \Gamma_{za}, \quad (\text{B.3.16})$$

we find that there is only one AdS-AdS integrability condition,

$$\left(\Xi_{\pm}^2 \pm \frac{1}{\ell}\Xi_{\pm}\right)\epsilon_{\pm} = 0. \quad (\text{B.3.17})$$

However,

$$\Xi_{\pm}^2 = \left[\mp \frac{1}{2\ell} + \frac{1}{2}\not\partial A \Gamma_z\right] \left[\mp \frac{1}{2\ell} + \frac{1}{2}\not\partial A \Gamma_z\right] \quad (\text{B.3.18})$$

$$= \frac{1}{4\ell^2} \mp \frac{1}{2\ell}\not\partial A \Gamma_z - \frac{1}{4}(dA)^2 \quad (\text{B.3.19})$$

$$\Xi_{\pm}^2 \pm \frac{1}{\ell}\Xi_{\pm} = -\frac{1}{4\ell^2} - \frac{1}{4}(dA)^2, \quad (\text{B.3.20})$$

so the integrability condition cannot be satisfied for $\ell < \infty$ or $dA \neq 0$.

B.4 Heterotic $\mathbb{R}^{1,n-1} \times M^{10-n}$

$\mathbb{R}^{1,n-1}$ spaces are very similar to AdS_n spaces. Again, the fields are purely magnetic, and the Bianchi identity is

$$dH = 0 \quad (\text{B.4.1})$$

while the field equations are

$$\nabla^2 \Phi = 2(d\Phi)^2 - \frac{1}{12}H^2 \quad (\text{B.4.2})$$

$$\nabla^k H_{ijk} = 2\partial^k \Phi H_{ijk}, \quad (\text{B.4.3})$$

and the Einstein equation includes only a transverse component,

$$R_{ij}^{(10-n)} = \frac{1}{4}H_{ik_1k_2}H_j{}^{k_1k_2} - 2\nabla_i \nabla_j \Phi, \quad (\text{B.4.4})$$

which contracts to

$$R^{(10-n)} = \frac{1}{4}H^2 - 2\nabla^2 \Phi \quad (\text{B.4.5})$$

$$= \frac{5}{12}H^2 - 4(d\Phi)^2. \quad (\text{B.4.6})$$

B.4.1 Killing Spinor Equations

The $\mathbb{R}^{1,n-1}$ -direction parallel transport equation is

$$\partial_{\mu}\epsilon = 0, \quad (\text{B.4.7})$$

so the Killing spinors have no dependence on these coordinates.

The parallel transport equations in the transverse directions are $\nabla^{(\pm)}\epsilon = \nabla_i\epsilon + \Psi_i^{(\pm)}\epsilon = 0$, where

$$\Psi_i^{(\pm)} = \frac{1}{8}\not\Gamma_i \Gamma_{11}, \quad (\text{B.4.8})$$

and the algebraic equation is $\mathcal{A}^{(\pm)}\epsilon_{\pm} = 0$, where

$$\mathcal{A}^{(\pm)} = \not\partial \Phi + \frac{1}{12}\not\Gamma \Gamma_{11}. \quad (\text{B.4.9})$$

B.4.2 Maximum Condition on ϵ_+

We introduce a new operator,

$$\hat{\nabla}_i^{(+)} = \nabla_i^{(+)} + q\Gamma_i\mathcal{A}^{(+)}, \quad (\text{B.4.10})$$

with the intention to demonstrate that, for an appropriately chosen value of q , if $\Gamma^i\hat{\nabla}_i^{(+)}\epsilon_+ = 0$, then ϵ_+ satisfies the Killing spinor equations. The modified Dirac condition is

$$\Gamma^i\hat{\nabla}_i^{(+)}\epsilon_+ = \left(\Gamma^i\nabla_i + \Gamma^i\Psi_i^{(+)} + (10-n)q\mathcal{A}^{(+)}\right)\epsilon_+ = 0. \quad (\text{B.4.11})$$

The Laplacian expands into two terms,

$$\nabla^2\|\epsilon_+\|^2 = 2\|\nabla\epsilon_+\|^2 + 2\langle\epsilon_+, \nabla^2\epsilon_+\rangle. \quad (\text{B.4.12})$$

The first term is then

$$\begin{aligned} 2\|\nabla\epsilon_+\|^2 &= 2\left\|\hat{\nabla}^{(+)}\epsilon_+\right\|^2 - 4\left\langle\epsilon_+, \left(\Psi^{(+)\dagger} + q\mathcal{A}^{(+)\dagger}\Gamma^i\right)\nabla_i\epsilon_+\right\rangle \\ &\quad - 2\left\langle\epsilon_+, \left(\Psi^{(+)\dagger} + q\mathcal{A}^{(+)\dagger}\Gamma^i\right)\left(\Psi_i^{(+)} + q\Gamma_i\mathcal{A}^{(+)}\right)\epsilon_+\right\rangle \\ &= 2\left\|\hat{\nabla}^{(+)}\epsilon_+\right\|^2 - 4\left\langle\epsilon_+, \Psi^{(+)\dagger}\nabla_i\epsilon_+\right\rangle \\ &\quad - 2\left\langle\epsilon_+, \left(\Psi^{(+)\dagger} - q\mathcal{A}^{(+)\dagger}\Gamma^i\right)\left(\Psi_i^{(+)} + q\Gamma_i\mathcal{A}^{(+)}\right)\epsilon_+\right\rangle, \end{aligned}$$

while the second term is

$$\begin{aligned} 2\langle\epsilon_+, \nabla^2\epsilon_+\rangle &= 2\langle\epsilon_+, \Gamma^i\nabla_i(\Gamma^j\nabla_j\epsilon_+)\rangle + \frac{1}{2}R^{(10-n)}\|\epsilon_+\|^2 \\ &= \frac{1}{2}R^{(10-n)}\|\epsilon_+\|^2 - 2\left\langle\epsilon_+, \nabla_i\left(\Gamma^i\Gamma^j\Psi_j^{(+)} + q(10-n)\Gamma^i\mathcal{A}^{(+)}\right)\epsilon_+\right\rangle \\ &\quad - 2\left\langle\epsilon_+, \left(\Gamma^i\Gamma^j\Psi_j^{(+)} + q(10-n)\Gamma^i\mathcal{A}^{(+)}\right)\nabla_i\epsilon_+\right\rangle. \end{aligned} \quad (\text{B.4.13})$$

Thus, the full expansion is

$$\begin{aligned} \nabla^2\|\epsilon_+\|^2 &= 2\left\|\hat{\nabla}^{(+)}\epsilon_+\right\|^2 + \frac{1}{2}R^{(10-n)}\|\epsilon_+\|^2 \\ &\quad + \left\langle\epsilon_+, \left[-4\Psi^{(+)\dagger} - 2\Gamma^i\Gamma^j\Psi_j^{(+)} - 2(10-n)q\Gamma^i\mathcal{A}\right]\nabla_i\epsilon_+\right\rangle \\ &\quad + \left\langle\epsilon_+, -2\left(\Psi^{(+)\dagger} - q\mathcal{A}^{(+)\dagger}\right)\left(\Psi_i^{(+)} + q\Gamma_i\mathcal{A}^{(+)}\right)\epsilon_+\right\rangle \\ &\quad + \left\langle\epsilon_+, \nabla_i\left[-2\Gamma^i\Gamma^j\Psi_j^{(+)} - 2(10-n)q\Gamma^i\mathcal{A}\right]\epsilon_+\right\rangle \end{aligned} \quad (\text{B.4.14})$$

where

$$\Psi_i^{(+)\dagger} = -\frac{1}{8}\not{H}_i\Gamma_{11} \quad (\text{B.4.15})$$

$$\mathcal{A}^{(+)\dagger} = \not{\Phi} + \frac{1}{12}\not{H}\Gamma_{11} \quad (\text{B.4.16})$$

We would like to write the third term in the form $\alpha^i \nabla_i \|\epsilon_+\|^2 + \langle \epsilon_+, \mathcal{F} \Gamma^i \tilde{\nabla}_i \epsilon_+ \rangle$. Expanded, this term is

$$\begin{aligned} & \left\langle \epsilon_+, \left[-4\Psi^{(+i)\dagger} - 2\Gamma^i \Gamma^j \Psi_j^{(+)} - 2(10-n)q\Gamma^i \mathcal{A} \right] \nabla_i \epsilon_+ \right\rangle \\ &= \left\langle \epsilon_+, \left[-2(10-n)q\Gamma^i \not{\partial} \Phi - \frac{1+2(10-n)q}{4} \not{H}^i \Gamma_{11} - \frac{3+2(10-n)q}{12} \Gamma \not{H}^i \Gamma_{11} \right] \nabla_i \epsilon_+ \right\rangle. \end{aligned} \quad (\text{B.4.17})$$

This fixes $q = -\frac{1}{10-n}$, so term is

$$\begin{aligned} & \left\langle \epsilon_+, \left[-4\Psi^{(+i)\dagger} - 2\Gamma^i \Gamma^j \Psi_j^{(+)} - 2(n-2)A^{-1}\Gamma^z \mathbb{B}^{(+)} + 2\Gamma^i \mathcal{A} \right] \nabla_i \epsilon_+ \right\rangle \\ &= \left\langle \epsilon_+, \left[2\Gamma_i \not{\partial} \Phi - \frac{1}{12} (\Gamma \not{H})_i \Gamma_{11} + \frac{1}{4} \not{H}_i \Gamma_{11} \right] \nabla^i \epsilon_+ \right\rangle, \end{aligned} \quad (\text{B.4.18})$$

which means

$$\mathcal{F} = -2\not{\partial} \Phi - \frac{1}{12} \not{H} \Gamma_{11} \quad (\text{B.4.19})$$

and $\alpha_i = 2\partial^i \Phi$.

Combining this with the fourth term in (B.8.6), we find

$$\begin{aligned} & \left\langle \epsilon_+, -2 \left(\Psi^{(+i)\dagger} + \frac{1}{10-n} \mathcal{A}^{(+i)\dagger} \Gamma^i + \frac{1}{2} \mathcal{F} \Gamma^i \right) \left(\Psi_i^{(+)} - \frac{1}{10-n} \Gamma_i \mathcal{A}^{(+)} \right) \epsilon_+ \right\rangle \\ &= \left\langle \epsilon_+, -2 \left[-\frac{9-n}{10-n} \not{\partial} \Phi \Gamma^i - \frac{1}{4(10-n)} \not{H}^i \Gamma_{11} - \frac{8-n}{24(10-n)} \Gamma \not{H}^i \Gamma_{11} \right] \right. \\ & \quad \times \left[-\frac{1}{10-n} \Gamma_i \not{\partial} \Phi + \frac{8-n}{8(10-n)} \not{H}_i \Gamma_{11} - \frac{1}{12(10-n)} \Gamma \not{H}_i \Gamma_{11} \right] \epsilon_+ \left. \right\rangle \\ &= \left\langle \epsilon_+, \left[-\frac{2(9-n)}{10-n} (d\Phi)^2 + \frac{1}{3(10-n)} \partial_i \Phi \Gamma \not{H}^i \Gamma_{11} - \frac{1}{72(10-n)} \not{H} \not{H} - \frac{1}{24} H^2 \right] \epsilon_+ \right\rangle \end{aligned} \quad (\text{B.4.20})$$

Using the field equations and Bianchi identities, the last term is

$$\left\langle \epsilon_+, \nabla_i \left[-2\Gamma^i \Gamma^j \Psi_j^{(+)} + 2\Gamma^i \mathcal{A} \right] \epsilon_+ \right\rangle = \left\langle \epsilon_+, \left[2\nabla^2 \Phi - \frac{1}{48} d \not{H} \Gamma_{11} \right] \epsilon_+ \right\rangle \quad (\text{B.4.21})$$

$$= \left\langle \epsilon_+, \left[4(d\Phi)^2 - \frac{1}{6} H^2 \right] \epsilon_+ \right\rangle \quad (\text{B.4.22})$$

while the curvature term is

$$\frac{1}{2} R^{(10-n)} \|\epsilon_+\|^2 = \left\langle \epsilon_+, \left[-2(d\Phi)^2 + \frac{5}{24} H^2 \right] \epsilon_+ \right\rangle. \quad (\text{B.4.23})$$

The sum of (B.7.40), (B.7.42), and (B.7.43), i.e. the second through fifth terms of (B.8.6), is

$$\left\langle \epsilon_+, \left[\frac{2}{10-n} (d\Phi)^2 + \frac{1}{3(10-n)} \partial_i \Phi \Gamma \not{H}^i \Gamma_{11} - \frac{1}{72(10-n)} \not{H} \not{H} \right] \epsilon_+ \right\rangle.$$

Comparing this to

$$\|\mathcal{A}\epsilon_+\|^2 = \left\langle \epsilon_+, \left[(d\Phi)^2 + \frac{1}{6} \partial_i \Phi (\Gamma \not{H})^i \Gamma_{11} - \frac{1}{144} \not{H} \not{H} \right] \epsilon_+ \right\rangle$$

we find that

$$\nabla^2 \|\epsilon_+\|^2 - 2\partial_i \Phi \nabla^i \|\epsilon_+\|^2 = 2 \left\| \hat{\nabla}^{(+)} \epsilon_+ \right\|^2 + \frac{2}{10-n} \left\| \mathcal{A}^{(+)} \epsilon_+ \right\|^2. \quad (\text{B.4.24})$$

B.5 IIA AdS₃

B.5.1 Fields

For AdS₃ backgrounds, the four-form field, G , and the three-form field, H , have electric components,

$$G = Ae^+ \wedge e^- \wedge dz \wedge X + Y \quad (\text{B.5.1})$$

$$H = AW e^+ \wedge e^- \wedge dz + Z. \quad (\text{B.5.2})$$

The remaining fields are purely magnetic.

B.5.2 Field equations and Bianchi Identities

The Bianchi identities are

$$dZ = 0 \quad (\text{B.5.3})$$

$$d(A^3 W) = 0 \quad (\text{B.5.4})$$

$$dS = Sd\Phi \quad (\text{B.5.5})$$

$$dF = d\Phi \wedge F + SZ + ASW e^+ \wedge e^- \wedge dz \quad (\text{B.5.6})$$

$$dY = d\Phi \wedge Y + Z \wedge F \quad (\text{B.5.7})$$

$$dX = -3A^{-1}dA \wedge X + d\Phi \wedge X - WF. \quad (\text{B.5.8})$$

From the magnetic part of (B.5.6), we see that either $S = 0$, or $W = 0$. The field equations are

$$\nabla^2 \Phi = -3A^{-1} \partial_i A \partial^i \Phi + 2(d\Phi)^2 - \frac{1}{12} Z^2 + \frac{1}{2} W^2 + \frac{5}{4} S^2 + \frac{3}{8} F^2 + \frac{1}{96} Y^2 - \frac{1}{4} X^2 \quad (\text{B.5.9})$$

$$\nabla^k H_{ijk} = -3A^{-1} \partial^k A H_{ijk} + 2\partial^k \Phi H_{ijk} + \frac{1}{2} Y_{ijk\ell} F^{k\ell} + S F_{ij} \quad (\text{B.5.10})$$

$$\nabla^j F_{ij} = -3A^{-1} \partial^j F_{ij} + \partial^j \Phi F_{ij} - W X_i - \frac{1}{6} Y_{ijk\ell} Z^{j\ell} \quad (\text{B.5.11})$$

$$\nabla^i X_i = \partial_i \Phi X^i + *_7(Z \wedge Y) \quad (\text{B.5.12})$$

$$\nabla^\ell G_{ijk\ell} = -3A^{-1} \partial^\ell A G_{ijk\ell} + \partial^\ell \Phi G_{ijk\ell} + *_7(WY - Z \wedge X)_{ijk} \quad (\text{B.5.13})$$

and the Einstein equation separates into an AdS component,

$$\nabla^2 \ln A = -\frac{2}{\ell^2} A^{-2} - \frac{3}{\ell^2} A^{-2} (dA)^2 + 2A^{-1} \partial_i A \partial^i \Phi + \frac{1}{2} W^2 + \frac{1}{4} S^2 + \frac{1}{8} F^2 + \frac{1}{96} Y^2 + \frac{1}{4} X^2 \quad (\text{B.5.14})$$

and a transverse component, which contracts to

$$R^{(7)} = 3\nabla^2 \ln A + 3A^{-2} (dA)^2 + \frac{1}{4} Z^2 - \frac{7}{4} S^2 - \frac{3}{8} F^2 + \frac{1}{96} Y^2 + \frac{5}{4} X^2 - 2\nabla^2 \Phi \quad (\text{B.5.15})$$

$$\begin{aligned} &= -\frac{6}{\ell^2} A^{-2} - 6A^{-2} (dA)^2 + 12A^{-1} \partial_i A \partial^i \Phi - 4(d\Phi)^2 + \frac{5}{12} Z^2 + \frac{1}{2} W^2 \\ &\quad - \frac{7}{2} S^2 - \frac{3}{4} F^2 + \frac{1}{48} Y^2 + \frac{5}{2} X^2 \end{aligned} \quad (\text{B.5.16})$$

B.5.3 Killing Spinor Equations

The AdS-direction parallel transport equations are

$$0 = \partial_u \epsilon_{\pm} + A^{-1} \Gamma_{+z} (\ell^{-1} - \Xi_{-}) \epsilon_{\mp} \quad (\text{B.5.17})$$

$$0 = \partial_r \epsilon_{\pm} - A^{-1} \Gamma_{-z} \Xi_{+} \epsilon_{\mp} \quad (\text{B.5.18})$$

$$0 = \partial_z \epsilon_{\pm} - \Xi_{\pm} \epsilon_{\pm} + \frac{2r}{\ell} A^{-1} \Gamma_{-z} \Xi_{+} \epsilon_{\mp} \quad (\text{B.5.19})$$

where, for AdS₃,

$$\Xi_{\pm} = \mp \frac{1}{2\ell} + \frac{1}{2} \not\partial A \Gamma_z \pm \frac{1}{4} A W \Gamma_{11} - \frac{1}{8} A S \Gamma_z - \frac{1}{16} A \not{F} \Gamma_z \Gamma_{11} - \frac{1}{192} A \not{Y} \Gamma_z \mp \frac{1}{8} A \not{X}. \quad (\text{B.5.20})$$

Note that for larger AdS dimensions, some of these fields will be identically zero.

Because

$$\Xi_{\pm} \Gamma_{z+} + \Gamma_{z+} \Xi_{\mp} = 0 \quad (\text{B.5.21})$$

$$\Xi_{\pm} \Gamma_{z-} + \Gamma_{z-} \Xi_{\mp} = 0 \quad (\text{B.5.22})$$

we find that there is only one AdS-AdS integrability condition,

$$(\Xi_{\pm}^2 \pm \ell^{-1} \Xi_{\pm}) \epsilon_{\pm} = 0. \quad (\text{B.5.23})$$

Thus, we can easily integrate over z , finding

$$\epsilon_{\pm}(0, 0, z, 0, y^i) = \sigma_{\pm}(y^i) + e^{\mp z/\ell} \tau_{\pm}(y^i), \quad (\text{B.5.24})$$

where

$$\Xi_{\pm} \sigma_{\pm} = 0 \quad \Xi_{\pm} \tau_{\pm} = \mp \ell^{-1} \tau_{\pm}. \quad (\text{B.5.25})$$

For convenience, we introduce $\mathbb{B}^{(\pm)}$, which represents Ξ_{\pm} when it acts on σ_{\pm} and $\Xi_{\pm} \pm 1$ when it acts on τ_{\pm} . The integrability condition is then succinctly expressed as $\mathbb{B}^{(\pm)} \chi_{\pm} = 0$, $\chi_{\pm} = \sigma_{\pm}, \tau_{\pm}$. Specifically,

$$\mathbb{B}^{(\pm)} = \mp \frac{c}{2\ell} + \frac{1}{2} \not\partial A \Gamma_z \pm \frac{1}{4} A W \Gamma_{11} - \frac{1}{8} A S \Gamma_z - \frac{1}{16} A \not{F} \Gamma_z \Gamma_{11} - \frac{1}{192} A \not{Y} \Gamma_z \mp \frac{1}{8} A \not{X}, \quad (\text{B.5.26})$$

where $c = 1$ when $\chi_{\pm} = \sigma_{\pm}$ and $c = -1$ when $\chi_{\pm} = \tau_{\pm}$.

The parallel transport equations in the transverse directions are $\nabla^{(\pm)} \epsilon = \nabla_i \epsilon + \Psi_i^{(\pm)} \epsilon = 0$, where

$$\Psi_i^{(\pm)} = \pm \frac{1}{2} A^{-1} \partial_i A + \frac{1}{8} \not{Z}_i \Gamma_{11} + \frac{1}{8} S \Gamma_i + \frac{1}{16} \not{F} \Gamma_i \Gamma_{11} + \frac{1}{192} \not{Y} \Gamma_i \pm \frac{1}{8} \not{X} \Gamma_{zi}, \quad (\text{B.5.27})$$

and the algebraic equation is $\mathcal{A} \epsilon_{\pm} = 0$, where

$$\mathcal{A}^{(\pm)} = \not\partial \Phi + \frac{1}{12} \not{Z} \Gamma_{11} \mp \frac{1}{2} W \Gamma_z \Gamma_{11} + \frac{5}{4} S + \frac{3}{8} \not{F} \Gamma_{11} + \frac{1}{96} \not{Y} \pm \frac{1}{4} \not{X} \Gamma_z. \quad (\text{B.5.28})$$

Each of these applies to σ_{\pm} and τ_{\pm} individually.

B.5.4 Maximum Condition on σ_+ , τ_+ , when $S = 0$

We introduce a new operator,

$$\hat{\nabla}_i^{(+,q_1,q_2)} = \nabla_i^{(+)} + q_1 A^{-1} \Gamma_{zi} \mathbb{B}^{(+)} + q_2 \Gamma_i \mathcal{A}^{(+)}, \quad (\text{B.5.29})$$

with the intention to demonstrate that, for an appropriately chosen value of q_1 and q_2 , if $\Gamma^i \hat{\nabla}_i \chi_+ = 0$, then χ_+ satisfies the Killing spinor equations. For convenience, we also introduce an operator representing a general linear combination of the algebraic conditions,

$$\mathbb{A}^{(+,q_1,q_2)} = -q_1 A^{-1} \Gamma_z \mathbb{B}^{(+)} + q_2 \mathcal{A}^{(+)}, \quad (\text{B.5.30})$$

so that $\hat{\nabla}_i^{(+,q_1,q_2)} = \nabla_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2)}$, and the modified Dirac condition is

$$\Gamma^i \hat{\nabla}_i^{(+,q_1,q_2)} \chi_+ = \left(\Gamma^i \nabla_i + \Gamma^i \Psi_i^{(+)} + 7 \mathbb{A}^{(+,q_1,q_2)} \right) \chi_+ = 0. \quad (\text{B.5.31})$$

The Laplacian expands into two terms,

$$\nabla^2 \|\chi_+\|^2 = 2 \|\nabla \chi_+\|^2 + 2 \langle \chi_+, \nabla^2 \chi_+ \rangle. \quad (\text{B.5.32})$$

The first term is then

$$\begin{aligned} 2 \|\nabla \chi_+\|^2 &= 2 \left\| \hat{\nabla}^{(+,q_1,q_2)} \chi_+ \right\|^2 - 4 \left\langle \chi_+, \left(\Psi^{(+)\dagger} + \mathbb{A}^{(+,q_1,q_2)\dagger} \Gamma^i \right) \nabla_i \chi_+ \right\rangle \\ &\quad - 2 \left\langle \chi_+, \left(\Psi^{(+)\dagger} + \mathbb{A}^{(+,q_1,q_2)\dagger} \Gamma^i \right) \left(\Psi_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2)} \right) \chi_+ \right\rangle \\ &= 2 \left\| \hat{\nabla}^{(+,q_1,q_2)} \chi_+ \right\|^2 - 4 \left\langle \chi_+, \Psi^{(+)\dagger} \nabla_i \chi_+ \right\rangle \\ &\quad - 2 \left\langle \chi_+, \left(\Psi^{(+)\dagger} - \mathbb{A}^{(+,q_1,q_2)\dagger} \Gamma^i \right) \left(\Psi_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2)} \right) \chi_+ \right\rangle, \end{aligned}$$

while the second term is

$$\begin{aligned} 2 \langle \chi_+, \nabla^2 \chi_+ \rangle &= 2 \langle \chi_+, \Gamma^i \nabla_i (\Gamma^j \nabla_j \chi_+) \rangle + \frac{1}{2} R^{(7)} \|\chi_+\|^2 \\ &= \frac{1}{2} R^{(7)} \|\chi_+\|^2 - 2 \left\langle \chi_+, \nabla_i \left(\Gamma^i \Gamma^j \Psi_j^{(+)} + 7 \Gamma^i \mathbb{A}^{(+,q_1,q_2)} \right) \chi_+ \right\rangle \\ &\quad - 2 \left\langle \chi_+, \left(\Gamma^i \Gamma^j \Psi_j^{(+)} + 7 \Gamma^i \mathbb{A}^{(+,q_1,q_2)} \right) \nabla_i \chi_+ \right\rangle. \end{aligned} \quad (\text{B.5.33})$$

Thus, the full expansion is

$$\begin{aligned} \nabla^2 \|\chi_+\|^2 &= 2 \left\| \hat{\nabla}^{(+,q_1,q_2)} \chi_+ \right\|^2 + \frac{1}{2} R^{(7)} \|\chi_+\|^2 \\ &\quad + \left\langle \chi_+, \left[-4 \Psi^{(+)\dagger} - 2 \Gamma^i \Gamma^j \Psi_j^{(+)} - 14 q_1 A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} - 14 q_2 \Gamma^i \mathcal{A}^{(+)} \right] \nabla_i \chi_+ \right\rangle \\ &\quad + \left\langle \chi_+, -2 \left(\Psi^{(+)\dagger} - \mathbb{A}^{(+,q_1,q_2)\dagger} \Gamma^i \right) \left(\Psi_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2)} \right) \chi_+ \right\rangle \\ &\quad + \left\langle \chi_+, \nabla_i \left[-2 \Gamma^i \Gamma^j \Psi_j^{(+)} - 14 q_1 A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} - 14 q_2 \Gamma^i \mathcal{A}^{(+)} \right] \chi_+ \right\rangle \end{aligned} \quad (\text{B.5.34})$$

where

$$\Psi_i^{(+)\dagger} = \frac{1}{2}A^{-1}\partial_i A - \frac{1}{8}\not{Z}_i\Gamma_{11} + \frac{1}{16}\Gamma_i\not{F}\Gamma_{11} + \frac{1}{192}\Gamma_i\not{Y} - \frac{1}{8}\Gamma_i\not{X}\Gamma_z \quad (\text{B.5.35})$$

$$\mathbb{B}^{(+)\dagger} = -\frac{c}{2\ell} - \frac{1}{2}\not{\partial}A\Gamma_z + \frac{1}{4}AW\Gamma_{11} - \frac{1}{16}A\not{F}\Gamma_z\Gamma_{11} - \frac{1}{192}A\not{Y}\Gamma_z - \frac{1}{8}A\not{X} \quad (\text{B.5.36})$$

$$\mathcal{A}^{(+)\dagger} = \not{\partial}\Phi + \frac{1}{12}\not{Z}\Gamma_{11} + \frac{1}{2}W\Gamma_z\Gamma_{11} - \frac{3}{8}\not{F}\Gamma_{11} + \frac{1}{96}\not{Y} - \frac{1}{4}\not{X}\Gamma_z. \quad (\text{B.5.37})$$

We would like to write the third term in the form $\alpha^i\nabla_i\|\chi_+\|^2 + \langle\chi_+, \mathcal{F}\Gamma^i\tilde{\nabla}_i\chi_+\rangle$. Expanded, this term is

$$\begin{aligned} & \left\langle \chi_+, \left[-4\Psi^{(+)\dagger} - 2\Gamma^i\Gamma^j\Psi_j^{(+)} - 14q_1A^{-1}\Gamma^{zi}\mathbb{B}^{(+)} - 14q_2\Gamma^i\mathcal{A}^{(+)} \right] \nabla_i\chi_+ \right\rangle \\ &= \left\langle \chi_+, \left[\frac{7q_1c}{\ell}A^{-1}\Gamma^{zi} - [3+7q_1]A^{-1}\partial^i A - [1+7q_1]A^{-1}(\Gamma\not{\partial}A)^i \right. \right. \\ & \quad - 14q_2\Gamma^i\not{\partial}\Phi - \frac{1+14q_2}{4}\not{Z}^i\Gamma_{11} - \frac{3+14q_2}{12}\Gamma\not{Z}^i\Gamma_{11} \\ & \quad - \frac{7q_1+14q_2}{2}W\Gamma^{zi}\Gamma_{11} - \frac{5+7q_1+42q_2}{8}\Gamma^i\not{F}\Gamma_{11} \\ & \quad \left. + \frac{-1-7q_1-14q_2}{96}\Gamma^i\not{Y} \right] \nabla_i\chi_+ \right\rangle. \end{aligned} \quad (\text{B.5.38})$$

This fixes $q_2 = -\frac{1}{7}$ and $q_1 = \frac{1}{7}$. The term is thus

$$\begin{aligned} & \left\langle \chi_+, \left[-4\Psi^{(+)\dagger} - 2\Gamma^i\Gamma^j\Psi_j^{(+)} - 2A^{-1}\Gamma^{zi}\mathbb{B}^{(+)} + 2\Gamma^i\mathcal{A} \right] \nabla_i\chi_+ \right\rangle \\ &= \left\langle \chi_+, \left[\frac{c}{\ell}A^{-1}\Gamma_{zi} - 4A^{-1}\partial_i A - 2A^{-1}(\Gamma\not{\partial}A)_i + 2\Gamma_i\not{\partial}\Phi \right. \right. \\ & \quad \left. \left. - \frac{1}{12}(\Gamma\not{Z})_i\Gamma_{11} + \frac{1}{4}\not{Z}_i\Gamma_{11} + \frac{1}{2}W\Gamma^{zi}\Gamma_{11} \right] \nabla^i\chi_+ \right\rangle, \end{aligned} \quad (\text{B.5.39})$$

so

$$\mathcal{F} = \frac{c}{\ell}A^{-1}\Gamma_z + 2A^{-1}\not{\partial}A - 2\not{\partial}\Phi - \frac{1}{12}\not{Z}\Gamma_{11} - \frac{1}{2}W\Gamma_z\Gamma_{11} \quad (\text{B.5.40})$$

and $\alpha_i = -3A^{-1}\partial^i A + 2\partial^i\Phi$.

Combining this with the fourth term in (B.5.34), we find

$$\begin{aligned}
& \left\langle \chi_+, -2 \left(\Psi^{(+)\dagger} + \frac{1}{7} A^{-1} \mathbb{B}^{(+)\dagger} \Gamma^{zi} + \frac{1}{7} \mathcal{A}^{(+)\dagger} \Gamma^i + \frac{1}{2} \mathcal{F} \Gamma^i \right) \left(\Psi_i^{(+)} + \frac{1}{7} A^{-1} \Gamma_{zi} \mathbb{B}^{(+)} - \frac{1}{7} \Gamma_i \mathcal{A}^{(+)} \right) \chi_+ \right\rangle \\
&= \left\langle \chi_+, -2 \left[\frac{3c}{7\ell} A^{-1} \Gamma^{zi} + \frac{10}{7} A^{-1} \partial^i A - \frac{13}{14} A^{-1} \Gamma \not\partial^i A - \frac{6}{7} \not\partial \Phi \Gamma^i - \frac{1}{28} \not{Z}^i \Gamma_{11} - \frac{5}{168} \Gamma \not{Z}^i \Gamma_{11} \right. \right. \\
&\quad \left. \left. + \frac{3}{14} W \Gamma^{zi} \Gamma_{11} + \frac{3}{28} \Gamma \not{F}^i \Gamma_{11} + \frac{1}{28} \not{F}^i \Gamma_{11} + \frac{1}{168} \Gamma \not{Y}^i + \frac{1}{56} \not{Y}^i \right. \right. \\
&\quad \left. \left. - \frac{5}{28} \Gamma \not{X}^i \Gamma_z - \frac{1}{14} X^i \Gamma_z \right] \right. \\
&\quad \times \left[-\frac{c}{14\ell} A^{-1} \Gamma_{zi} + \frac{4}{7} A^{-1} \partial_i A + \frac{1}{14} A^{-1} \Gamma \not\partial_i A - \frac{1}{7} \Gamma_i \not\partial \Phi + \frac{5}{56} \not{Z}_i \Gamma_{11} - \frac{1}{84} \Gamma \not{Z}_i \Gamma_{11} \right. \\
&\quad \left. - \frac{1}{28} W \Gamma_{zi} \Gamma_{11} + \frac{1}{56} \Gamma \not{F}_i \Gamma_{11} - \frac{3}{14} \not{F}_i \Gamma_{11} + \frac{1}{224} \Gamma \not{Y}_i - \frac{1}{42} \not{Y}_i \right. \\
&\quad \left. \left. + \frac{1}{14} \Gamma \not{X}_i \Gamma_z - \frac{5}{28} X_i \Gamma_z \right] \chi_+ \right\rangle \tag{B.5.41}
\end{aligned}$$

$$\begin{aligned}
&= \left\langle \chi_+, \left[-\frac{3}{7\ell^2} A^{-2} - \frac{17}{7} A^{-2} (dA)^2 + \frac{26}{7} A^{-1} \partial^i A \partial_i \Phi - \frac{12}{7} (d\Phi)^2 + \frac{c}{42\ell} A^{-1} \not{Z} \Gamma_z \Gamma_{11} \right. \right. \\
&\quad - \frac{1}{42} A^{-1} \partial_i A \Gamma \not{Z}^i \Gamma_{11} + \frac{1}{21} \partial_i \Phi \Gamma \not{Z}^i \Gamma_{11} - \frac{1}{504} \not{Z} \not{Z} - \frac{1}{24} Z^2 - \frac{3c}{7\ell} A^{-1} W \Gamma_{11} \\
&\quad + \frac{1}{84} W \not{Z} \Gamma_z \Gamma_{11} - \frac{3}{28} W^2 + \frac{c}{28\ell} A^{-1} \not{F} \Gamma_z \Gamma_{11} + \frac{1}{28} A^{-1} \partial_i A \Gamma \not{F}^i \Gamma_{11} \\
&\quad - \frac{1}{14} \partial_i \Phi \Gamma \not{F}^i \Gamma_{11} - \frac{3}{112} \not{Z} \not{F} + \frac{1}{8} \not{Z}^i \not{F}^i - \frac{1}{28} \not{F} \not{F} - \frac{1}{8} F^2 + \frac{c}{112\ell} A^{-1} \not{Y} \Gamma_z \\
&\quad + \frac{1}{112} A^{-1} \partial_i A \Gamma \not{Y}^i - \frac{1}{56} \partial_i \Phi \Gamma \not{Y}^i - \frac{13}{42 \cdot 96} \not{Z} \not{Y} \Gamma_{11} + \frac{1}{48} \not{Z}_i \not{Y}^i \Gamma_{11} - \frac{1}{448} \not{F} \not{Y} \Gamma_{11} \\
&\quad + \frac{1}{168 \cdot 96} \not{Y} \not{Y} - \frac{1}{96} Y^2 + \frac{5c}{14\ell} A^{-1} \not{X} - \frac{5}{14} A^{-1} X_i \partial^i A \Gamma_z + \frac{5}{7} X_i \partial^i \Phi \Gamma_z \\
&\quad \left. \left. + \frac{3}{56} X_i \not{Z}^i \Gamma_z \Gamma_{11} - \frac{1}{28} X_i \Gamma \not{F}^i \Gamma_z \Gamma_{11} + \frac{1}{168} X_i \not{Y}^i \Gamma_z - \frac{5}{28} X^2 \right] \chi_+ \right\rangle \tag{B.5.42}
\end{aligned}$$

Using the field equations and Bianchi identities, the last term is

$$\begin{aligned}
& \left\langle \chi_+, \nabla_i \left[-2\Gamma^i \Gamma^j \Psi_j^{(+)} - 2A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} + 2\Gamma^i \mathcal{A} \right] \chi_+ \right\rangle \\
&= \left\langle \chi_+, \left[-2\nabla^2 \ln A + 2\nabla^2 \Phi - \frac{1}{48} d\not{Z} \Gamma_{11} + \frac{1}{12} d\not{F} \Gamma_{11} + \frac{1}{240} d\not{Y} - \frac{1}{2} \nabla_i X^i \Gamma_z \right] \chi_+ \right\rangle \tag{B.5.43}
\end{aligned}$$

$$\begin{aligned}
&= \left\langle \chi_+, \left[\frac{4}{\ell^2} A^{-2} + 6A^{-2} (dA)^2 - 10A^{-1} \partial^i A \partial_i \Phi + 4(d\Phi)^2 - \frac{1}{6} Z^2 + \frac{1}{4} \partial_i \Phi \Gamma \not{F}^i \Gamma_{11} \right. \right. \\
&\quad \left. \left. + \frac{1}{24} \not{Z} \not{F} - \frac{1}{8} \not{Z}^i \not{F}_i + \frac{1}{2} F^2 + \frac{1}{48} \partial_i \Phi \Gamma \not{Y}^i \right. \right. \\
&\quad \left. \left. + \frac{1}{288} \not{Z} \not{Y} \Gamma_{11} - \frac{1}{48} \not{Z}_i \not{Y}^i \Gamma_{11} - \frac{1}{2} X^i \partial_i \Phi \Gamma_z - X^2 \right] \chi_+ \right\rangle \tag{B.5.44}
\end{aligned}$$

while the curvature term is

$$\begin{aligned} \frac{1}{2}R^{(10-n)}\|\chi_+\|^2 = & \left\langle \chi_+, \left[-\frac{3}{\ell^2}A^{-2} - 3A^{-2}(dA)^2 + 6A^{-1}\partial^i A\partial_i\Phi - 2(d\Phi)^2 \right. \right. \\ & \left. \left. + \frac{5}{24}Z^2 + \frac{1}{4}W^2 - \frac{3}{8}F^2 + \frac{1}{96}Y^2 + \frac{5}{4}X^2 \right] \chi_+ \right\rangle. \end{aligned} \quad (\text{B.5.45})$$

The sum of (B.5.42), (B.5.44), and (B.5.45), i.e. the second through fifth terms of (B.5.34), is

$$\begin{aligned} & \left\langle \chi_+, \left[\frac{4}{7\ell^2}A^{-2} + \frac{4}{7}A^{-2}(dA)^2 - \frac{2}{7}A^{-1}\partial^i A\partial_i\Phi + \frac{2}{7}(d\Phi)^2 + \frac{c}{42\ell}A^{-1}\not{Z}\Gamma_z\Gamma_{11} \right. \right. \\ & - \frac{1}{42}A^{-1}\partial_i A\Gamma\not{Z}^i\Gamma_{11} + \frac{1}{21}\partial_i\Phi\Gamma\not{Z}^i\Gamma_{11} - \frac{1}{504}\not{Z}\not{Z} - \frac{3c}{7\ell}A^{-1}W\Gamma_{11} \\ & + \frac{1}{84}W\not{Z}\Gamma_z\Gamma_{11} + \frac{1}{7}W^2 + \frac{c}{28\ell}A^{-1}\not{F}\Gamma_z\Gamma_{11} + \frac{1}{28}A^{-1}\partial_i A\Gamma\not{F}^i\Gamma_{11} \\ & + \frac{5}{28}\partial_i\Phi\Gamma\not{F}^i\Gamma_{11} + \frac{5}{336}\not{Z}\not{F} - \frac{1}{28}\not{F}\not{F} + \frac{c}{112\ell}A^{-1}\not{Y}\Gamma_z + \frac{1}{112}A^{-1}\partial_i A\Gamma\not{Y}^i \\ & + \frac{1}{336}\partial_i\Phi\Gamma\not{Y}^i + \frac{1}{42\cdot 96}\not{Z}\not{Y}\Gamma_{11} - \frac{1}{448}\not{F}\not{Y}\Gamma_{11} + \frac{1}{168\cdot 96}\not{Y}\not{Y} \\ & + \frac{5c}{14\ell}A^{-1}\not{X} - \frac{5}{14}A^{-1}X_i\partial^i A\Gamma_z + \frac{3}{14}X_i\partial^i\Phi\Gamma_z \\ & \left. \left. + \frac{3}{56}X_i\not{Z}^i\Gamma_z\Gamma_{11} - \frac{1}{28}X_i\Gamma\not{F}^i\Gamma_z\Gamma_{11} + \frac{1}{168}X_i\not{Y}^i\Gamma_z + \frac{1}{14}X^2 \right] \chi_+ \right\rangle \end{aligned} \quad (\text{B.5.46})$$

Comparing this to

$$\begin{aligned} \left\| \mathbb{B}^{(+)} \chi_+ \right\|^2 = & \left\langle \chi_+, \left[\frac{1}{4\ell^2} + \frac{1}{4}(dA)^2 - \frac{c}{4\ell} AW\Gamma_{11} + \frac{1}{16} A^2 W^2 + \frac{c}{16\ell} A \not{F} \Gamma_z \Gamma_{11} \right. \right. \\ & + \frac{1}{16} A \partial_i A \Gamma \not{F}^i - \frac{1}{256} A^2 \not{F} \not{F} + \frac{c}{192\ell} A \not{Y} \Gamma_z + \frac{1}{192} A \partial_i A \Gamma \not{Y}^i \\ & - \frac{1}{16 \cdot 96} \not{F} \not{Y} \Gamma_{11} + \frac{1}{192^2} A^2 \not{Y} \not{Y} + \frac{c}{8\ell} A \not{X} - \frac{1}{8} A X^i \partial_i A \Gamma_z \\ & \left. \left. + \frac{1}{64} A^2 X_i \Gamma \not{F}^i \Gamma_z \Gamma_{11} + \frac{1}{192} A^2 X_i \not{Y}^i \Gamma_z + \frac{1}{64} A^2 X^2 \right] \chi_+ \right\rangle \end{aligned} \quad (\text{B.5.47})$$

$$\begin{aligned} \left\langle \Gamma_z \mathbb{B}^{(+)} \chi_+, \mathcal{A} \chi_+ \right\rangle = & \left\langle \chi_+, \left[-\frac{1}{2} \partial^i A \partial_i \Phi + \frac{c}{24\ell} \not{Z} \Gamma_z \Gamma_{11} - \frac{1}{24} \partial_i A \Gamma \not{Z}^i \Gamma_{11} + \frac{c}{4\ell} W \Gamma_{11} \right. \right. \\ & - \frac{1}{48} A W \not{Z} \Gamma_z - \frac{1}{8} A W^2 - \frac{3c}{16\ell} \not{F} \Gamma_z \Gamma_{11} - \frac{3}{16} \partial_i A \Gamma \not{F}^i \Gamma_{11} \\ & - \frac{1}{16} A \partial_i \Phi \Gamma \not{F}^i \Gamma_{11} - \frac{A}{192} \not{Z} \not{F} + \frac{3}{128} A \not{F} \not{F} - \frac{c}{192\ell} \not{Y} \Gamma_z \\ & - \frac{1}{192} \partial_i A \Gamma \not{Y}^i - \frac{1}{192} A \partial_i \Phi \Gamma \not{Y}^i - \frac{1}{24 \cdot 96} A \not{Z} \not{Y} \Gamma_{11} \\ & + \frac{1}{384} A \not{F} \not{Y} \Gamma_{11} - \frac{1}{96 \cdot 192} A \not{Y} \not{Y} + \frac{c}{8\ell} \not{X} - \frac{1}{8} X^i \partial_i A \Gamma_z \\ & + \frac{1}{8} A X^i \partial_i \Phi \Gamma_z + \frac{1}{32} A X_i \not{Z}^i \Gamma_z \Gamma_{11} \\ & \left. \left. - \frac{1}{32} A X_i \Gamma \not{F}^i \Gamma_z \Gamma_{11} + \frac{1}{32} A X^2 \right] \chi_+ \right\rangle \end{aligned} \quad (\text{B.5.48})$$

$$\begin{aligned} \left\| \mathcal{A}^{(+)} \chi_+ \right\|^2 = & \left\langle \chi_+, \left[(d\Phi)^2 + \frac{1}{6} \partial_i \Phi \Gamma \not{Z}^i \Gamma_{11} - \frac{1}{144} \not{Z} \not{Z} + \frac{1}{12} W \not{Z} \Gamma_z + \frac{1}{4} W^2 \right. \right. \\ & + \frac{3}{4} \partial_i \Phi \Gamma \not{F}^i \Gamma_{11} + \frac{1}{16} \not{Z} \not{F} - \frac{9}{64} \not{F} \not{F} + \frac{1}{48} \partial_i \Phi \Gamma \not{Y}^i \\ & + \frac{1}{576} \not{Z} \not{Y} \Gamma_{11} - \frac{1}{128} \not{F} \not{Y} \Gamma_{11} + \frac{1}{96^2} \not{Y} \not{Y} + \frac{1}{2} X^i \partial_i \Phi \Gamma_z \\ & \left. \left. + \frac{1}{8} X_i \not{Z}^i \Gamma_z \Gamma_{11} - \frac{3}{16} X_i \Gamma \not{F}^i \Gamma_z \Gamma_{11} - \frac{1}{48} X_i \not{Y}^i \Gamma_z + \frac{1}{16} X^2 \right] \chi_+ \right\rangle \end{aligned} \quad (\text{B.5.49})$$

we find that

$$\begin{aligned} & \nabla^2 \|\chi_+\|^2 + (3A^{-1} \partial_i A - 2\partial_i \Phi) \nabla^i \|\chi_+\|^2 \\ & = \left\| \hat{\nabla} \chi_+ \right\|^2 + \frac{16}{7} A^{-2} \left\| \mathbb{B}^{(+)} \chi_+ \right\|^2 + \frac{4}{7} A^{-1} \left\langle \Gamma_z \mathbb{B}^{(+)} \chi_+, \mathcal{A} \chi_+ \right\rangle + \frac{2}{7} \|\mathcal{A} \chi_+\|^2. \end{aligned} \quad (\text{B.5.50})$$

B.5.5 Maximum Condition on σ_+ , τ_+ , when $W = 0$

We introduce a new operator,

$$\hat{\nabla}_i^{(+, q_1, q_2)} = \nabla_i^{(+)} + q_1 A^{-1} \Gamma_{zi} \mathbb{B}^{(+)} + q_2 \Gamma_i \mathcal{A}^{(+)}, \quad (\text{B.5.51})$$

with the intention to demonstrate that, for an appropriately chosen value of q_1 and q_2 , if $\Gamma^i \hat{\nabla}_i \chi_+ = 0$, then χ_+ satisfies the Killing spinor equations. For convenience, we also intro-

duce an operator representing a general linear combination of the algebraic conditions,

$$\mathbb{A}^{(+,q_1,q_2)} = -q_1 A^{-1} \Gamma_z \mathbb{B}^{(+)} + q_2 \mathcal{A}^{(+)}, \quad (\text{B.5.52})$$

so that $\hat{\nabla}_i^{(+,q_1,q_2)} = \nabla_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2)}$, and the modified Dirac condition is

$$\Gamma^i \hat{\nabla}_i^{(+,q_1,q_2)} \chi_+ = \left(\Gamma^i \nabla_i + \Gamma^i \Psi_i^{(+)} + 6 \mathbb{A}^{(+,q_1,q_2)} \right) \chi_+ = 0. \quad (\text{B.5.53})$$

The Laplacian expands into two terms,

$$\nabla^2 \|\chi_+\|^2 = 2 \|\nabla \chi_+\|^2 + 2 \langle \chi_+, \nabla^2 \chi_+ \rangle. \quad (\text{B.5.54})$$

The first term is then

$$\begin{aligned} 2 \|\nabla \chi_+\|^2 &= 2 \left\| \hat{\nabla}^{(+,q_1,q_2)} \chi_+ \right\|^2 - 4 \left\langle \chi_+, \left(\Psi^{(+i)\dagger} + \mathbb{A}^{(+,q_1,q_2)\dagger} \Gamma^i \right) \nabla_i \chi_+ \right\rangle \\ &\quad - 2 \left\langle \chi_+, \left(\Psi^{(+i)\dagger} + \mathbb{A}^{(+,q_1,q_2)\dagger} \Gamma^i \right) \left(\Psi_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2)} \right) \chi_+ \right\rangle \\ &= 2 \left\| \hat{\nabla}^{(+,q_1,q_2)} \chi_+ \right\|^2 - 4 \left\langle \chi_+, \Psi^{(+i)\dagger} \nabla_i \chi_+ \right\rangle \\ &\quad - 2 \left\langle \chi_+, \left(\Psi^{(+i)\dagger} - \mathbb{A}^{(+,q_1,q_2)\dagger} \Gamma^i \right) \left(\Psi_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2)} \right) \chi_+ \right\rangle, \end{aligned}$$

while the second term is

$$\begin{aligned} 2 \langle \chi_+, \nabla^2 \chi_+ \rangle &= 2 \langle \chi_+, \Gamma^i \nabla_i (\Gamma^j \nabla_j \chi_+) \rangle + \frac{1}{2} R^{(7)} \|\chi_+\|^2 \\ &= \frac{1}{2} R^{(7)} \|\chi_+\|^2 - 2 \left\langle \chi_+, \nabla_i \left(\Gamma^i \Gamma^j \Psi_j^{(+)} + 7 \Gamma^i \mathbb{A}^{(+,q_1,q_2)} \right) \chi_+ \right\rangle \\ &\quad - 2 \left\langle \chi_+, \left(\Gamma^i \Gamma^j \Psi_j^{(+)} + 7 \Gamma^i \mathbb{A}^{(+,q_1,q_2)} \right) \nabla_i \chi_+ \right\rangle. \end{aligned} \quad (\text{B.5.55})$$

Thus, the full expansion is

$$\begin{aligned} \nabla^2 \|\chi_+\|^2 &= 2 \left\| \hat{\nabla}^{(+,q_1,q_2)} \chi_+ \right\|^2 + \frac{1}{2} R^{(7)} \|\chi_+\|^2 \\ &\quad + \left\langle \chi_+, \left[-4 \Psi^{(+i)\dagger} - 2 \Gamma^i \Gamma^j \Psi_j^{(+)} - 14 q_1 A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} - 14 q_2 \Gamma^i \mathcal{A}^{(+)} \right] \nabla_i \chi_+ \right\rangle \\ &\quad + \left\langle \chi_+, -2 \left(\Psi^{(+i)\dagger} - \mathbb{A}^{(+,q_1,q_2)\dagger} \Gamma^i \right) \left(\Psi_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2)} \right) \chi_+ \right\rangle \\ &\quad + \left\langle \chi_+, \nabla_i \left[-2 \Gamma^i \Gamma^j \Psi_j^{(+)} - 14 q_1 A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} - 14 q_2 \Gamma^i \mathcal{A}^{(+)} \right] \chi_+ \right\rangle \end{aligned} \quad (\text{B.5.56})$$

where

$$\Psi_i^{(+)\dagger} = \frac{1}{2} A^{-1} \partial_i A - \frac{1}{8} \not{Z}_i \Gamma_{11} + \frac{1}{8} S \Gamma_i + \frac{1}{16} \Gamma_i \not{F} \Gamma_{11} + \frac{1}{192} \Gamma_i \not{Y} - \frac{1}{8} \Gamma_i \not{X} \Gamma_z \quad (\text{B.5.57})$$

$$\mathbb{B}^{(+)\dagger} = -\frac{c}{2\ell} - \frac{1}{2} \not{\partial} A \Gamma_z - \frac{1}{8} A S \Gamma_z - \frac{1}{16} A \not{F} \Gamma_z \Gamma_{11} - \frac{1}{192} A \not{Y} \Gamma_z - \frac{1}{8} A \not{X} \quad (\text{B.5.58})$$

$$\mathcal{A}^{(+)\dagger} = \not{\partial} \Phi + \frac{1}{12} \not{Z} \Gamma_{11} + \frac{5}{4} S - \frac{3}{8} \not{F} \Gamma_{11} + \frac{1}{96} \not{Y} - \frac{1}{4} \not{X} \Gamma_z. \quad (\text{B.5.59})$$

We would like to write the third term in the form $\alpha^i \nabla_i \|\chi_+\|^2 + \langle \chi_+, \mathcal{F} \Gamma^i \tilde{\nabla}_i \chi_+ \rangle$. Expanded, this term is

$$\begin{aligned}
& \left\langle \chi_+, \left[-4\Psi^{(+i)\dagger} - 2\Gamma^i \Gamma^j \Psi_j^{(+)} - 14q_1 A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} - 14q_2 \Gamma^i \mathcal{A}^{(+)} \right] \nabla_i \chi_+ \right\rangle \\
&= \left\langle \chi_+, \left[\frac{(10-n)q_1 c}{\ell} A^{-1} \Gamma^{zi} - [3+7q_1] A^{-1} \partial^i A - [1+7q_1] A^{-1} (\Gamma \not{A})^i \right. \right. \\
&\quad \left. \left. - 14q_2 \Gamma^i \not{\partial} \Phi - \frac{1+14q_2}{4} \not{Z}^i \Gamma_{11} - \frac{3+14q_2}{12} \Gamma \not{Z}^i \Gamma_{11} \right. \right. \\
&\quad \left. \left. - \frac{9+7q_1+70q_2}{4} S \Gamma^i - \frac{5+7q_1+42q_2}{8} \Gamma^i \not{F} \Gamma_{11} \right. \right. \\
&\quad \left. \left. + \frac{-1-7q_1-14q_2}{96} \Gamma^i \not{Y} \right] \nabla_i \chi_+ \right\rangle.
\end{aligned} \tag{B.5.60}$$

This fixes $q_2 = -\frac{1}{7}$ and $q_1 = \frac{1}{7}$. The term is thus

$$\begin{aligned}
& \left\langle \chi_+, \left[-4\Psi^{(+i)\dagger} - 2\Gamma^i \Gamma^j \Psi_j^{(+)} - 2A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} + 2\Gamma^i \mathcal{A} \right] \nabla_i \chi_+ \right\rangle \\
&= \left\langle \chi_+, \left[\frac{c}{\ell} A^{-1} \Gamma_{zi} - 4A^{-1} \partial_i A - 2A^{-1} (\Gamma \not{A})_i + 2\Gamma_i \not{\partial} \Phi \right. \right. \\
&\quad \left. \left. - \frac{1}{12} (\Gamma \not{Z})_i \Gamma_{11} + \frac{1}{4} \not{Z}_i \Gamma_{11} \right] \nabla^i \chi_+ \right\rangle,
\end{aligned} \tag{B.5.61}$$

so

$$\mathcal{F} = \frac{c}{\ell} A^{-1} \Gamma_z + 2A^{-1} \not{\partial} A - 2\not{\partial} \Phi - \frac{1}{12} \not{Z} \Gamma_{11} \tag{B.5.62}$$

and $\alpha_i = -3A^{-1} \partial^i A + 2\partial^i \Phi$.

Combining this with the fourth term in (B.5.56), we find

$$\begin{aligned}
& \left\langle \chi_+, -2 \left(\Psi^{(+)\dagger i} + \frac{1}{7} A^{-1} \mathbb{B}^{(+)\dagger} \Gamma^{zi} + \frac{1}{7} \mathcal{A}^{(+)\dagger} \Gamma^i + \frac{1}{2} \mathcal{F} \Gamma^i \right) \left(\Psi_i^{(+)} + \frac{1}{7} A^{-1} \Gamma_{zi} \mathbb{B}^{(+)} - \frac{1}{7} \Gamma_i \mathcal{A}^{(+)} \right) \chi_+ \right\rangle \\
&= \left\langle \chi_+, -2 \left[\frac{3c}{7\ell} A^{-1} \Gamma^{zi} + \frac{10}{7} A^{-1} \partial^i A - \frac{13}{14} A^{-1} \Gamma \not\partial^i A - \frac{6}{7} \not\partial \Phi \Gamma^i - \frac{1}{28} \not{Z}^i \Gamma_{11} - \frac{5}{168} \Gamma \not{Z}^i \Gamma_{11} \right. \right. \\
&\quad \left. \left. + \frac{2}{7} S \Gamma^i + \frac{3}{28} \Gamma \not{F}^i \Gamma_{11} + \frac{1}{28} \not{F}^i \Gamma_{11} + \frac{1}{168} \Gamma \not{Y}^i + \frac{1}{56} \not{Y}^i \right. \right. \\
&\quad \left. \left. - \frac{5}{28} \Gamma \not{X}^i \Gamma_z - \frac{1}{14} X^i \Gamma_z \right] \right. \\
&\quad \times \left[-\frac{c}{14\ell} A^{-1} \Gamma_{zi} + \frac{4}{7} A^{-1} \partial_i A + \frac{1}{14} A^{-1} \Gamma \not\partial_i A - \frac{1}{7} \Gamma_i \not\partial \Phi + \frac{5}{56} \not{Z}_i \Gamma_{11} - \frac{1}{84} \Gamma \not{Z}_i \Gamma_{11} \right. \\
&\quad \left. - \frac{1}{28} S \Gamma_i + \frac{1}{56} \Gamma \not{F}_i \Gamma_{11} - \frac{3}{14} \not{F}_i \Gamma_{11} + \frac{1}{224} \Gamma \not{Y}_i - \frac{1}{42} \not{Y}_i \right. \\
&\quad \left. \left. + \frac{1}{14} \Gamma \not{X}_i \Gamma_z - \frac{5}{28} X_i \Gamma_z \right] \chi_+ \right\rangle \tag{B.5.63}
\end{aligned}$$

$$\begin{aligned}
&= \left\langle \chi_+, \left[-\frac{3}{7\ell^2} A^{-2} - \frac{17}{7} A^{-2} (dA)^2 + \frac{26}{7} A^{-1} \partial^i A \partial_i \Phi - \frac{12}{7} (d\Phi)^2 + \frac{c}{42\ell} A^{-1} \not{Z} \Gamma_z \Gamma_{11} \right. \right. \\
&\quad - \frac{1}{42} A^{-1} \partial_i A \Gamma \not{Z}^i \Gamma_{11} + \frac{1}{21} \partial_i \Phi \Gamma \not{Z}^i \Gamma_{11} - \frac{1}{504} \not{Z} \not{Z} - \frac{1}{24} Z^2 - \frac{c}{14\ell} A^{-1} S \Gamma_z \\
&\quad - \frac{1}{14} A^{-1} S \not\partial A + \frac{1}{7} S \not\partial \Phi - \frac{5}{168} S \not{Z} \Gamma_{11} + \frac{1}{7} S^2 + \frac{c}{28\ell} A^{-1} \not{F} \Gamma_z \Gamma_{11} \\
&\quad + \frac{1}{28} A^{-1} \partial_i A \Gamma \not{F}^i \Gamma_{11} - \frac{1}{14} \partial_i \Phi \Gamma \not{F}^i \Gamma_{11} - \frac{3}{112} \not{Z} \not{F} + \frac{1}{8} \not{Z}^i \not{F}^i - \frac{1}{28} \not{F} \not{F} - \frac{1}{8} F^2 \\
&\quad + \frac{c}{112\ell} A^{-1} \not{Y} \Gamma_z + \frac{1}{112} A^{-1} \partial_i A \Gamma \not{Y}^i - \frac{1}{56} \partial_i \Phi \Gamma \not{Y}^i - \frac{13}{42 \cdot 96} \not{Z} \not{Y} \Gamma_{11} \\
&\quad + \frac{1}{48} \not{Z}_i \not{Y}^i \Gamma_{11} + \frac{1}{168} S \not{Y} - \frac{1}{448} \not{F} \not{Y} \Gamma_{11} + \frac{1}{168 \cdot 96} \not{Y} \not{Y} - \frac{1}{96} Y^2 \\
&\quad + \frac{5c}{14\ell} A^{-1} \not{X} - \frac{5}{14} A^{-1} X_i \partial^i A \Gamma_z + \frac{5}{7} X_i \partial^i \Phi \Gamma_z + \frac{3}{56} X_i \not{Z}^i \Gamma_z \Gamma_{11} \\
&\quad \left. \left. - \frac{1}{28} X_i \Gamma \not{F}^i \Gamma_z \Gamma_{11} + \frac{1}{168} X_i \not{Y}^i \Gamma_z - \frac{5}{28} X^2 \right] \chi_+ \right\rangle \tag{B.5.64}
\end{aligned}$$

Using the field equations and Bianchi identities, the last term is

$$\begin{aligned}
& \left\langle \chi_+, \nabla_i \left[-2\Gamma^i \Gamma^j \Psi_j^{(+)} - 2A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} + 2\Gamma^i \mathcal{A} \right] \chi_+ \right\rangle \\
&= \left\langle \chi_+, \left[-2\nabla^2 \ln A + 2\nabla^2 \Phi - \frac{1}{48} d\mathbb{Z} \Gamma_{11} + \frac{1}{2} \not\partial S + \frac{1}{12} d\mathbb{F} \Gamma_{11} + \frac{1}{240} dX - \frac{1}{2} \nabla_i X^i \Gamma_z \right] \chi_+ \right\rangle \tag{B.5.65}
\end{aligned}$$

$$\begin{aligned}
&= \left\langle \chi_+, \left[\frac{4}{\ell^2} A^{-2} + 6A^{-2} (dA)^2 - 10A^{-1} \partial^i A \partial_i \Phi + 4(d\Phi)^2 - \frac{1}{6} Z^2 + \frac{1}{2} S \not\partial \Phi \right. \right. \\
&\quad + \frac{1}{12} S \not{Z} \Gamma_{11} + 2S^2 + \frac{1}{4} \partial_i \Phi \Gamma \not{F}^i \Gamma_{11} + \frac{1}{24} \not{Z} \not{F} - \frac{1}{8} \not{Z}^i \not{F}_i \\
&\quad + \frac{1}{2} F^2 + \frac{1}{48} \partial_i \Phi \Gamma \not{Y}^i + \frac{1}{288} \not{Z} \not{Y} \Gamma_{11} \\
&\quad \left. \left. - \frac{1}{48} \not{Z}_i \not{Y}^i \Gamma_{11} - \frac{1}{2} X^i \partial_i \Phi \Gamma_z - X^2 \right] \chi_+ \right\rangle \tag{B.5.66}
\end{aligned}$$

while the curvature term is

$$\begin{aligned} \frac{1}{2}R^{(10-n)}\|\chi_+\|^2 = & \left\langle \chi_+, \left[-\frac{3}{\ell^2}A^{-2} - 3A^{-2}(dA)^2 + 6A^{-1}\partial^i A\partial_i\Phi - 2(d\Phi)^2 \right. \right. \\ & \left. \left. + \frac{5}{24}Z^2 - \frac{7}{4}S^2 - \frac{3}{8}F^2 + \frac{1}{96}Y^2 + \frac{5}{4}X^2 \right] \chi_+ \right\rangle. \end{aligned} \quad (\text{B.5.67})$$

The sum of (B.5.64), (B.5.66), and (B.5.67), i.e. the second through fifth terms of (B.5.56), is

$$\begin{aligned} & \left\langle \chi_+, \left[\frac{4}{7\ell^2}A^{-2} + \frac{4}{7}A^{-2}(dA)^2 - \frac{2}{7}A^{-1}\partial^i A\partial_i\Phi + \frac{2}{7}(d\Phi)^2 + \frac{c}{42\ell}A^{-1}\not{Z}\Gamma_z\Gamma_{11} \right. \right. \\ & - \frac{1}{42}A^{-1}\partial_i A\Gamma\not{Z}^i\Gamma_{11} + \frac{1}{21}\partial_i\Phi\Gamma\not{Z}^i\Gamma_{11} - \frac{1}{504}\not{Z}\not{Z} - \frac{c}{14\ell}A^{-1}S\Gamma_z \\ & - \frac{1}{14}A^{-1}S\not{\partial}A + \frac{9}{14}S\not{\partial}\Phi + \frac{3}{56}S\not{Z}\Gamma_{11} + \frac{11}{28}S^2 + \frac{c}{28\ell}A^{-1}\not{F}\Gamma_z\Gamma_{11} \\ & + \frac{1}{28}A^{-1}\partial_i A\Gamma\not{F}^i\Gamma_{11} + \frac{5}{28}\partial_i\Phi\Gamma\not{F}^i\Gamma_{11} + \frac{5}{336}\not{Z}\not{F} - \frac{1}{28}\not{F}\not{F} \\ & + \frac{c}{112\ell}A^{-1}\not{Y}\Gamma_z + \frac{1}{112}A^{-1}\partial_i A\Gamma\not{Y}^i + \frac{1}{336}\partial_i\Phi\Gamma\not{Y}^i + \frac{1}{42\cdot 96}\not{Z}\not{Y}\Gamma_{11} \\ & + \frac{1}{168}S\not{Y} - \frac{1}{448}\not{F}\not{Y}\Gamma_{11} + \frac{1}{168\cdot 96}\not{Y}\not{Y} + \frac{5c}{14\ell}A^{-1}\not{X} - \frac{5}{14}A^{-1}X_i\partial^i A\Gamma_z \\ & \left. \left. + \frac{3}{14}X_i\partial^i\Phi\Gamma_z + \frac{3}{56}X_i\not{Z}^i\Gamma_z\Gamma_{11} - \frac{1}{28}X_i\Gamma\not{F}^i\Gamma_z\Gamma_{11} + \frac{1}{168}X_i\not{Y}^i\Gamma_z + \frac{1}{14}X^2 \right] \chi_+ \right\rangle \end{aligned} \quad (\text{B.5.68})$$

Comparing this to

$$\begin{aligned} \left\| \mathbb{B}^{(+)} \chi_+ \right\|^2 = & \left\langle \chi_+, \left[\frac{1}{4\ell^2} + \frac{1}{4}(dA)^2 + \frac{c}{8\ell} AS\Gamma_z + \frac{1}{8} AS\partial A + \frac{1}{64} A^2 S^2 + \frac{c}{16\ell} A\# \Gamma_z \Gamma_{11} \right. \right. \\ & + \frac{1}{16} A\partial_i A \Gamma \#^i - \frac{1}{256} A^2 \# \# + \frac{c}{192\ell} A\dot{Y} \Gamma_z + \frac{1}{192} A\partial_i A \Gamma \dot{Y}^i \\ & + \frac{1}{768} A^2 S\dot{Y} - \frac{1}{16 \cdot 96} \# \dot{Y} \Gamma_{11} + \frac{1}{192^2} A^2 \dot{Y} \dot{Y} + \frac{c}{8\ell} A\dot{X} \\ & \left. \left. - \frac{1}{8} A X^i \partial_i A \Gamma_z + \frac{1}{64} A^2 X_i \Gamma \#^i \Gamma_z \Gamma_{11} + \frac{1}{192} A^2 X_i \dot{Y}^i \Gamma_z + \frac{1}{64} A^2 X^2 \right] \chi_+ \right\rangle \end{aligned} \quad (\text{B.5.69})$$

$$\begin{aligned} \left\langle \Gamma_z \mathbb{B}^{(+)} \chi_+, \mathcal{A} \chi_+ \right\rangle = & \left\langle \chi_+, \left[-\frac{1}{2} \partial^i A \partial_i \Phi + \frac{c}{24\ell} \dot{Z} \Gamma_z \Gamma_{11} - \frac{1}{24} \partial_i A \Gamma \dot{Z}^i \Gamma_{11} - \frac{5c}{8\ell} S \Gamma_z \right. \right. \\ & - \frac{5}{8} S \partial A - \frac{1}{8} AS \partial \Phi - \frac{1}{96} AS \dot{Z} \Gamma_{11} - \frac{5}{32} AS^2 - \frac{3c}{16\ell} \# \Gamma_z \Gamma_{11} \\ & - \frac{3}{16} \partial_i A \Gamma \#^i \Gamma_{11} - \frac{1}{16} A \partial_i \Phi \Gamma \#^i \Gamma_{11} - \frac{A}{192} \dot{Z} \# + \frac{3}{128} A \# \# \\ & - \frac{c}{192\ell} \dot{Y} \Gamma_z - \frac{1}{192} \partial_i A \Gamma \dot{Y}^i - \frac{1}{192} A \partial_i \Phi \Gamma \dot{Y}^i - \frac{1}{24 \cdot 96} A \dot{Z} \dot{Y} \Gamma_{11} \\ & - \frac{1}{128} AS \dot{Y} + \frac{1}{384} A \# \dot{Y} \Gamma_{11} - \frac{1}{96 \cdot 192} A \dot{Y} \dot{Y} + \frac{c}{8\ell} \dot{X} \\ & - \frac{1}{8} X^i \partial_i A \Gamma_z + \frac{1}{8} A X^i \partial_i \Phi \Gamma_z + \frac{1}{32} A X_i \dot{Z}^i \Gamma_z \Gamma_{11} \\ & \left. \left. - \frac{1}{32} A X_i \Gamma \#^i \Gamma_z \Gamma_{11} + \frac{1}{32} A X^2 \right] \chi_+ \right\rangle \end{aligned} \quad (\text{B.5.70})$$

$$\begin{aligned} \left\| \mathcal{A}^{(+)} \chi_+ \right\|^2 = & \left\langle \chi_+, \left[(d\Phi)^2 + \frac{1}{6} \partial_i \Phi \Gamma \dot{Z}^i \Gamma_{11} - \frac{1}{144} \dot{Z} \dot{Z} + \frac{5}{2} S \partial \Phi + \frac{5}{24} S \dot{Z} \Gamma_{11} + \frac{25}{16} S^2 \right. \right. \\ & + \frac{3}{4} \partial_i \Phi \Gamma \#^i \Gamma_{11} + \frac{1}{16} \dot{Z} \# - \frac{9}{64} \# \# + \frac{1}{48} \partial_i \Phi \Gamma \dot{Y}^i \\ & + \frac{1}{576} \dot{Z} \dot{Y} \Gamma_{11} + \frac{5}{192} S \dot{Y} - \frac{1}{128} \# \dot{Y} \Gamma_{11} + \frac{1}{96^2} \dot{Y} \dot{Y} + \frac{1}{2} X^i \partial_i \Phi \Gamma_z \\ & \left. \left. + \frac{1}{8} X_i \dot{Z}^i \Gamma_z \Gamma_{11} - \frac{3}{16} X_i \Gamma \#^i \Gamma_z \Gamma_{11} - \frac{1}{48} X_i \dot{Y}^i \Gamma_z + \frac{1}{16} X^2 \right] \chi_+ \right\rangle \end{aligned} \quad (\text{B.5.71})$$

we find that

$$\begin{aligned} & \nabla^2 \|\chi_+\|^2 + (3A^{-1} \partial_i A - 2\partial_i \Phi) \nabla^i \|\chi_+\|^2 \\ & = \left\| \hat{\nabla} \chi_+ \right\|^2 + \frac{16}{7} A^{-2} \left\| \mathbb{B}^{(+)} \chi_+ \right\|^2 + \frac{4}{7} A^{-1} \left\langle \Gamma_z \mathbb{B}^{(+)} \chi_+, \mathcal{A} \chi_+ \right\rangle + \frac{2}{7} \|\mathcal{A} \chi_+\|^2. \end{aligned} \quad (\text{B.5.72})$$

B.6 IIA AdS₄

B.6.1 Fields

For AdS₄ backgrounds, the four-form field, G , has an electric component, corresponding to a scalar field on the transverse space,

$$G = A^2 e^+ \wedge e^- \wedge dz \wedge dx X + Y. \quad (\text{B.6.1})$$

The remaining fields are purely magnetic.

B.6.2 Field equations and Bianchi Identities

The Bianchi identities are

$$dH = 0 \quad (\text{B.6.2})$$

$$dS = Sd\Phi \quad (\text{B.6.3})$$

$$dF = d\Phi \wedge F + SH \quad (\text{B.6.4})$$

$$dY = d\Phi \wedge Y + H \wedge F \quad (\text{B.6.5})$$

$$d(A^4 X) = A^4 d\Phi, \quad (\text{B.6.6})$$

the field equations are

$$\nabla^2 \Phi = -4A^{-1} \partial^i A \partial_i \Phi + 2(d\Phi)^2 + \frac{5}{4}S^2 + \frac{3}{8}F^2 - \frac{1}{12}H^2 + \frac{1}{96}Y^2 - \frac{1}{4}X^2 \quad (\text{B.6.7})$$

$$\nabla^k H_{ijk} = -4A^{-1} \partial^k A H_{ijk} + 2\partial^k \Phi H_{ijk} + S F_{ij} + \frac{1}{2} F^{k\ell} G_{ijk\ell} \quad (\text{B.6.8})$$

$$\nabla^j F_{ij} = -4A^{-1} \partial^j A F_{ij} + \partial^j \Phi F_{ij} - \frac{1}{6} F^{jk\ell} G_{ijk\ell} \quad (\text{B.6.9})$$

$$\nabla^\ell Y_{ijk\ell} = -4A^{-1} \partial^\ell A Y_{ijk\ell} + \partial^\ell \Phi Y_{ijk\ell} \quad (\text{B.6.10})$$

and the Einstein equation separates into an AdS component,

$$\nabla^2 \ln A = \frac{1}{96}Y^2 + \frac{1}{4}X^2 + \frac{1}{4}S^2 + \frac{1}{8}F^2 - 3\ell^{-2}A^{-2} - 4A^{-2}(dA)^2, \quad (\text{B.6.11})$$

and a transverse component, which contracts to

$$R^{(6)} = 4\nabla^2 \ln A + 4A^{-2}(dA)^2 + \frac{1}{48}Y^2 + \frac{3}{2}X^2 - \frac{3}{2}S^2 + \frac{1}{4}H^2 - \frac{1}{4}F^2 - 2\nabla^2 \Phi \quad (\text{B.6.12})$$

$$\begin{aligned} &= -12\ell^{-2}A^{-2} - 12A^{-2}(dA)^2 + \frac{1}{24}Y^2 + 3X^2 - 3S^2 + \frac{5}{12}H^2 \\ &\quad - \frac{1}{2}F^2 + 8A^{-1}\partial_i A \partial^i \Phi - 4(d\Phi)^2. \end{aligned} \quad (\text{B.6.13})$$

B.6.3 Killing Spinor Equations

The AdS-direction parallel transport equations are

$$0 = \partial_u \epsilon_\pm + A^{-1} \Gamma_{+z} (\ell^{-1} - \Xi_-) \epsilon_\mp \quad (\text{B.6.14})$$

$$0 = \partial_r \epsilon_\pm - A^{-1} \Gamma_{-z} \Xi_+ \epsilon_\mp \quad (\text{B.6.15})$$

$$0 = \partial_z \epsilon_\pm - \Xi_\pm \epsilon_\pm + \frac{2r}{\ell} A^{-1} \Gamma_{-z} \Xi_+ \epsilon_\mp \quad (\text{B.6.16})$$

$$0 = \partial_x \epsilon_+ + e^{z/\ell} \Gamma_{zx} \Xi_+ \epsilon_+ \quad (\text{B.6.17})$$

$$0 = \partial_x \epsilon_- + e^{z/\ell} \Gamma_{zx} (\Xi_- - \ell^{-1}) \epsilon_- \quad (\text{B.6.18})$$

where, for AdS_4 ,

$$\Xi_{\pm} = \mp \frac{1}{2\ell} + \frac{1}{2}\not\partial A\Gamma_z - \frac{1}{8}A S\Gamma_z - \frac{1}{16}A\not{H}\Gamma_z\Gamma_{11} - \frac{1}{192}A\dot{Y}\Gamma_z \mp \frac{1}{8}AX\Gamma_x. \quad (\text{B.6.19})$$

Because

$$\Xi_{\pm}\Gamma_{z+} + \Gamma_{z+}\Xi_{\mp} = 0 \quad (\text{B.6.20})$$

$$\Xi_{\pm}\Gamma_{z-} + \Gamma_{z-}\Xi_{\mp} = 0 \quad (\text{B.6.21})$$

$$\Xi_{\pm}\Gamma_{zx} + \Gamma_{zx}\Xi_{\pm} = \mp \ell^{-1}\Gamma_{zx}, \quad (\text{B.6.22})$$

we find that there is only one AdS-AdS integrability condition,

$$(\Xi_{\pm}^2 \pm \ell^{-1}\Xi_{\pm})\epsilon_{\pm} = 0. \quad (\text{B.6.23})$$

Thus, we can easily integrate over z , finding

$$\epsilon_{\pm}(0, 0, z, 0, y^i) = \sigma_{\pm}(y^i) + e^{\mp z/\ell}\tau_{\pm}(y^i), \quad (\text{B.6.24})$$

where

$$\Xi_{\pm}\sigma_{\pm} = 0 \quad \Xi_{\pm}\tau_{\pm} = \mp \ell^{-1}\tau_{\pm}. \quad (\text{B.6.25})$$

For convenience, we introduce $\mathbb{B}^{(\pm)}$, which represents Ξ_{\pm} when it acts on σ_{\pm} and $\Xi_{\pm} \pm 1$ when it acts on τ_{\pm} . The integrability condition is then succinctly expressed as $\mathbb{B}^{(\pm)}\chi_{\pm} = 0$, $\chi_{\pm} = \sigma_{\pm}, \tau_{\pm}$. Specifically,

$$\mathbb{B}^{(\pm)} = \mp \frac{c}{2\ell} + \frac{1}{2}\not\partial A\Gamma_z - \frac{1}{8}A S\Gamma_z - \frac{1}{16}A\not{H}\Gamma_z\Gamma_{11} - \frac{1}{192}A\dot{Y}\Gamma_z \mp \frac{1}{8}AX\Gamma_x, \quad (\text{B.6.26})$$

where $c = 1$ when $\chi_{\pm} = \sigma_{\pm}$ and $c = -1$ when $\chi_{\pm} = \tau_{\pm}$.

The parallel transport equations in the transverse directions are $\nabla^{(\pm)}\epsilon = \nabla_i\epsilon + \Psi_i^{(\pm)}\epsilon = 0$, where

$$\Psi_i^{(\pm)} = \pm \frac{1}{2A}\partial_i A + \frac{1}{8}\not{H}_i\Gamma_{11} + \frac{1}{8}S\Gamma_i + \frac{1}{16}\not{H}\Gamma_i\Gamma_{11} + \frac{1}{192}\dot{Y}\Gamma_i \mp \frac{1}{8}X\Gamma_{zxi}, \quad (\text{B.6.27})$$

and the algebraic equation is $\mathcal{A}\epsilon_{\pm} = 0$, where

$$\mathcal{A} = \not\partial\Phi + \frac{1}{12}\not{H}\Gamma_{11} + \frac{5}{4}S + \frac{3}{8}\not{H}\Gamma_{11} + \frac{1}{96}\dot{Y} \mp \frac{1}{4}X\Gamma_{zx}. \quad (\text{B.6.28})$$

Each of these applies to σ_{\pm} and τ_{\pm} individually.

B.6.4 Maximum Condition on σ_+ , τ_+

We introduce a new operator,

$$\hat{\nabla}_i^{(+, q_1, q_2)} = \nabla_i^{(+)} + q_1 A^{-1}\Gamma_{zi}\mathbb{B}^{(+)} + q_2 \Gamma_i \mathcal{A}, \quad (\text{B.6.29})$$

with the intention to demonstrate that, for an appropriately chosen value of q_1 and q_2 , if $\Gamma^i \hat{\nabla}_i \chi_+ = 0$, then χ_+ satisfies the Killing spinor equations. For convenience, we also introduce an operator representing a general linear combination of the algebraic conditions,

$$\mathbb{A}^{(+,q_1,q_2)} = -q_1 A^{-1} \Gamma_z \mathbb{B}^{(+)} + q_2 \mathcal{A}, \quad (\text{B.6.30})$$

so that $\hat{\nabla}_i^{(+,q_1,q_2)} = \nabla_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2)}$, and the modified Dirac condition is

$$\Gamma^i \hat{\nabla}_i^{(+,q_1,q_2)} \chi_+ = \left(\Gamma^i \nabla_i + \Gamma^i \Psi_i^{(+)} + 6 \mathbb{A}^{(+,q_1,q_2)} \right) \chi_+ = 0. \quad (\text{B.6.31})$$

The Laplacian expands into two terms,

$$\nabla^2 \|\chi_+\|^2 = 2 \|\nabla \chi_+\|^2 + 2 \langle \chi_+, \nabla^2 \chi_+ \rangle. \quad (\text{B.6.32})$$

The first term is then

$$\begin{aligned} 2 \|\nabla \chi_+\|^2 &= 2 \left\| \hat{\nabla}^{(+,q_1,q_2)} \chi_+ \right\|^2 - 4 \left\langle \chi_+, \left(\Psi^{(+i)\dagger} + \mathbb{A}^{(+,q_1,q_2)\dagger} \Gamma^i \right) \nabla_i \chi_+ \right\rangle \\ &\quad - 2 \left\langle \chi_+, \left(\Psi^{(+i)\dagger} + \mathbb{A}^{(+,q_1,q_2)\dagger} \Gamma^i \right) \left(\Psi_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2)} \right) \chi_+ \right\rangle \\ &= 2 \left\| \hat{\nabla}^{(+,q_1,q_2)} \chi_+ \right\|^2 - 4 \left\langle \chi_+, \Psi^{(+i)\dagger} \nabla_i \chi_+ \right\rangle \\ &\quad - 2 \left\langle \chi_+, \left(\Psi^{(+i)\dagger} - \mathbb{A}^{(+,q_1,q_2)\dagger} \Gamma^i \right) \left(\Psi_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2)} \right) \chi_+ \right\rangle, \end{aligned}$$

while the second term is

$$\begin{aligned} 2 \langle \chi_+, \nabla^2 \chi_+ \rangle &= 2 \langle \chi_+, \Gamma^i \nabla_i (\Gamma^j \nabla_j \chi_+) \rangle + \frac{1}{2} R^{(6)} \|\chi_+\|^2 \\ &= \frac{1}{2} R^{(6)} \|\chi_+\|^2 - 2 \left\langle \chi_+, \nabla_i \left(\Gamma^i \Gamma^j \Psi_j^{(+)} + 6 \Gamma^i \mathbb{A}^{(+,q_1,q_2)} \right) \chi_+ \right\rangle \\ &\quad - 2 \left\langle \chi_+, \left(\Gamma^i \Gamma^j \Psi_j^{(+)} + 6 \Gamma^i \mathbb{A}^{(+,q_1,q_2)} \right) \nabla_i \chi_+ \right\rangle. \end{aligned} \quad (\text{B.6.33})$$

Thus, the full expansion is

$$\begin{aligned} \nabla^2 \|\chi_+\|^2 &= 2 \left\| \hat{\nabla}^{(+,q_1,q_2)} \chi_+ \right\|^2 + \frac{1}{2} R^{(6)} \|\chi_+\|^2 \\ &\quad + \left\langle \chi_+, \left[-4 \Psi^{(+i)\dagger} - 2 \Gamma^i \Gamma^j \Psi_j^{(+)} - 12 q_1 A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} - 12 q_2 \Gamma^i \mathcal{A} \right] \nabla_i \chi_+ \right\rangle \\ &\quad + \left\langle \chi_+, -2 \left(\Psi^{(+i)\dagger} - \mathbb{A}^{(+,q_1,q_2)\dagger} \Gamma^i \right) \left(\Psi_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2)} \right) \chi_+ \right\rangle \\ &\quad + \left\langle \chi_+, \nabla_i \left[-2 \Gamma^i \Gamma^j \Psi_j^{(+)} - 12 q_1 A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} - 12 q_2 \Gamma^i \mathcal{A} \right] \chi_+ \right\rangle \end{aligned} \quad (\text{B.6.34})$$

where

$$\Psi_i^{(+)\dagger} = \frac{1}{2} A^{-1} \partial_i A - \frac{1}{8} \not{H}_i \Gamma_{11} + \frac{1}{8} S \Gamma_i + \frac{1}{16} \Gamma_i \not{H} \Gamma_{11} + \frac{1}{192} \Gamma_i \not{Y} + \frac{1}{8} X \Gamma_{zxi} \quad (\text{B.6.35})$$

$$\mathbb{B}^{(+)\dagger} = -\frac{c}{2\ell} - \frac{1}{2} \not{\partial} A \Gamma_z - \frac{1}{8} A S \Gamma_z - \frac{1}{16} A \not{H} \Gamma_z \Gamma_{11} - \frac{1}{192} A \not{Y} \Gamma_z - \frac{1}{8} A X \Gamma_x \quad (\text{B.6.36})$$

$$\mathcal{A}^\dagger = \not{\partial} \Phi + \frac{5}{4} S + \frac{1}{12} \not{H} \Gamma_{11} - \frac{3}{8} \not{H} \Gamma_{11} + \frac{1}{96} \not{Y} + \frac{1}{4} X \Gamma_{zx}. \quad (\text{B.6.37})$$

We would like to write the third term in the form $\alpha^i \nabla_i \|\chi_+\|^2 + \langle \chi_+, \mathcal{F} \Gamma^i \tilde{\nabla}_i \chi_+ \rangle$. Expanded, this term is

$$\begin{aligned} & \left\langle \chi_+, \left[-4\Psi^{(+)\dagger i} - 2\Gamma^i \Gamma^j \Psi_j^{(+)} - 12q_1 A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} - 12q_2 \Gamma^i \mathcal{A} \right] \nabla_i \chi_+ \right\rangle \\ &= \left\langle \chi_+, \left[\frac{6q_1 c}{\ell} A^{-1} \Gamma^{zi} - [3 + 6q_1] A^{-1} \partial^i A - [1 + 6q_1] A^{-1} (\Gamma \not{\partial} A)^i - 12q_2 \Gamma^i \not{\partial} \Phi \right. \right. \\ &\quad \left. - \frac{8 + 6q_1 + 60q_2}{4} S \Gamma^i - \frac{1 + 12q_2}{4} \not{H}^i \Gamma_{11} - \frac{1 + 4q_2}{4} \Gamma \not{H}^i \Gamma_{11} \right. \\ &\quad \left. - \frac{2 + 3q_1 + 18q_2}{4} \Gamma^i \not{H} \Gamma_{11} - \frac{q_1 + 2q_2}{16} \Gamma^i \not{Y} + \frac{2 - 3q_1 + 6q_2}{2} X \Gamma^{zxi} \right] \nabla_i \chi_+ \right\rangle. \end{aligned} \quad (\text{B.6.38})$$

This fixes $q_2 = -\frac{1}{6}$ and $q_1 = \frac{1}{3}$. The term is thus

$$\begin{aligned} & \left\langle \chi_+, \left[-4\Psi^{(+)\dagger i} - 2\Gamma^i \Gamma^j \Psi_j^{(+)} - 4A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} + 2\Gamma^i \mathcal{A} \right] \nabla_i \chi_+ \right\rangle \\ &= \left\langle \chi_+, \left[\frac{2c}{\ell} A^{-1} \Gamma_{zi} - 5A^{-1} \partial_i A - 3A^{-1} \Gamma \not{\partial}_i A + 2\Gamma_i \not{\partial} \Phi \right. \right. \\ &\quad \left. \left. - \frac{1}{12} (\Gamma \not{H})_i \Gamma_{11} + \frac{1}{4} \not{H}_i \Gamma_{11} \right] \nabla^i \chi_+ \right\rangle, \end{aligned} \quad (\text{B.6.39})$$

so

$$\mathcal{F} = \frac{2c}{\ell} A^{-1} \Gamma_z + 3A^{-1} \not{\partial} A - 2\not{\partial} \Phi - \frac{1}{12} \not{H} \Gamma_{11} \quad (\text{B.6.40})$$

and $\alpha_i = -4A^{-1} \partial^i A + 2\partial^i \Phi$.

B.7 IIA AdS_n , $n \geq 5$

B.7.1 Field equations and Bianchi Identities

For AdS_n , $n \geq 5$, all fields are purely magnetic. The Bianchi identities are

$$dH = 0 \quad (\text{B.7.1})$$

$$dS = S d\Phi \quad (\text{B.7.2})$$

$$dF = d\Phi \wedge F + S H \quad (\text{B.7.3})$$

$$dG = d\Phi \wedge G + H \wedge F, \quad (\text{B.7.4})$$

the field equations are

$$\nabla^2 \Phi = -n A^{-1} \partial^i A \partial_i \Phi + 2(d\Phi)^2 + \frac{5}{4} S^2 + \frac{3}{8} F^2 - \frac{1}{12} H^2 + \frac{1}{96} G^2 \quad (\text{B.7.5})$$

$$\nabla^k H_{ijk} = -n A^{-1} \partial^k A H_{ijk} + 2\partial^k \Phi H_{ijk} + S F_{ij} + \frac{1}{2} F^{k\ell} G_{ijk\ell} \quad (\text{B.7.6})$$

$$\nabla^j F_{ij} = -n A^{-1} \partial^j A F_{ij} + \partial^j \Phi F_{ij} - \frac{1}{6} F^{jkl} G_{ijk\ell} \quad (\text{B.7.7})$$

$$\nabla^\ell G_{ijk\ell} = -n A^{-1} \partial^\ell A G_{ijk\ell} + \partial^\ell \Phi G_{ijk\ell} \quad (\text{B.7.8})$$

and the Einstein equation separates into an AdS component,

$$\nabla^2 \ln A = -(n-1)\ell^{-2}A^{-2} - nA^{-2}(dA)^2 + 2A^{-1}\partial_i A \partial^i \Phi + \frac{1}{96}G^2 + \frac{1}{4}S^2 + \frac{1}{8}F^2, \quad (\text{B.7.9})$$

and a transverse component, which contracts to

$$R^{(10-n)} = n\nabla^2 \ln A + nA^{-2}(dA)^2 + \frac{n-2}{96}G^2 - \frac{10-n}{4}S^2 + \frac{1}{4}H^2 + \frac{n-6}{8}F^2 - 2\nabla^2 \Phi \quad (\text{B.7.10})$$

$$\begin{aligned} &= -n(n-1)\ell^{-2}A^{-2} - n(n-1)A^{-2}(dA)^2 + \frac{n-2}{48}G^2 \\ &\quad - \frac{10-n}{2}S^2 + \frac{5}{12}H^2 + \frac{n-6}{4}F^2 + 4nA^{-1}\partial_i A \partial^i \Phi - 4(d\Phi)^2. \end{aligned} \quad (\text{B.7.11})$$

B.7.2 Killing Spinor Equations

The AdS-direction parallel transport equations are

$$0 = \partial_u \epsilon_{\pm} + A^{-1}\Gamma_{+z}(\ell^{-1} - \Xi_{\mp})\epsilon_{\mp} \quad (\text{B.7.12})$$

$$0 = \partial_r \epsilon_{\pm} - A^{-1}\Gamma_{-z}\Xi_{\pm}\epsilon_{\mp} \quad (\text{B.7.13})$$

$$0 = \partial_z \epsilon_{\pm} - \Xi_{\pm}\epsilon_{\pm} + \frac{2r}{\ell}A^{-1}\Gamma_{-z}\Xi_{\pm}\epsilon_{\mp} \quad (\text{B.7.14})$$

$$0 = \partial_a \epsilon_{+} + e^{z/\ell}\Gamma_{za}\Xi_{+}\epsilon_{+} \quad (\text{B.7.15})$$

$$0 = \partial_a \epsilon_{-} + e^{z/\ell}\Gamma_{za}(\Xi_{-} - \ell^{-1})\epsilon_{-} \quad (\text{B.7.16})$$

where, for AdS_k , $k \geq 5$,

$$\Xi_{\pm} = \mp \frac{1}{2\ell} + \frac{1}{2}\not\partial A \Gamma_z - \frac{A}{8}S \Gamma_z - \frac{A}{16}\not{F} \Gamma_z \Gamma_{11} - \frac{A}{192}\not{G} \Gamma_z. \quad (\text{B.7.17})$$

Note that for larger AdS dimensions, some of these fields will be identically zero.

Because

$$\Xi_{\pm}\Gamma_{z+} + \Gamma_{z+}\Xi_{\mp} = 0 \quad (\text{B.7.18})$$

$$\Xi_{\pm}\Gamma_{z-} + \Gamma_{z-}\Xi_{\mp} = 0 \quad (\text{B.7.19})$$

$$\Xi_{\pm}\Gamma_{za} + \Gamma_{za}\Xi_{\pm} = \mp \ell^{-1}\Gamma_{za}, \quad (\text{B.7.20})$$

we find that there is only one AdS-AdS integrability condition,

$$(\Xi_{\pm}^2 \pm \ell^{-1}\Xi_{\pm})\epsilon_{\pm} = 0. \quad (\text{B.7.21})$$

Thus, we can easily integrate over z , finding

$$\epsilon_{\pm}(0, 0, z, 0, y^i) = \sigma_{\pm}(y^i) + e^{\mp z/\ell}\tau_{\pm}(y^i), \quad (\text{B.7.22})$$

where

$$\Xi_{\pm}\sigma_{\pm} = 0 \quad \Xi_{\pm}\tau_{\pm} = \mp \ell^{-1}\tau_{\pm}. \quad (\text{B.7.23})$$

For convenience, we introduce $\mathbb{B}^{(\pm)}$, which represents Ξ_{\pm} when it acts on σ_{\pm} and $\Xi_{\pm} \pm 1$ when it acts on τ_{\pm} . The integrability condition is then succinctly expressed as $\mathbb{B}^{(\pm)}\chi_{\pm} = 0$, $\chi_{\pm} = \sigma_{\pm}, \tau_{\pm}$. Specifically,

$$\mathbb{B}^{(\pm)} = \mp \frac{c}{2\ell} + \frac{1}{2}\not\partial A\Gamma_z - \frac{1}{8}A S\Gamma_z - \frac{1}{16}A \not{F}\Gamma_z\Gamma_{11} - \frac{1}{192}A \not{G}\Gamma_z, \quad (\text{B.7.24})$$

where $c = 1$ when $\chi_{\pm} = \sigma_{\pm}$ and $c = -1$ when $\chi_{\pm} = \tau_{\pm}$.

The parallel transport equations in the transverse directions are $\nabla^{(\pm)}\epsilon = \nabla_i\epsilon + \Psi_i^{(\pm)}\epsilon = 0$, where

$$\Psi_i^{(\pm)} = \pm \frac{1}{2A}\partial_i A + \frac{1}{8}\not{H}_i\Gamma_{11} + \frac{1}{8}S\Gamma_i + \frac{1}{16}\not{F}\Gamma_i\Gamma_{11} + \frac{1}{192}\not{G}\Gamma_i, \quad (\text{B.7.25})$$

and the algebraic equation is $\mathcal{A}\epsilon_{\pm} = 0$, where

$$\mathcal{A}^{(\pm)} = \not\partial\Phi + \frac{1}{12}\not{H}\Gamma_{11} + \frac{5}{4}S + \frac{3}{8}\not{F}\Gamma_{11} + \frac{1}{96}\not{G}. \quad (\text{B.7.26})$$

Each of these applies to σ_{\pm} and τ_{\pm} individually.

B.7.3 Maximum Condition on σ_+ , τ_+

We introduce a new operator,

$$\hat{\nabla}_i^{(+,q_1,q_2)} = \nabla_i^{(+)} + q_1 A^{-1}\Gamma_z \mathbb{B}^{(+)} + q_2 \Gamma_i \mathcal{A}, \quad (\text{B.7.27})$$

with the intention to demonstrate that, for an appropriately chosen value of q_1 and q_2 , if $\Gamma^i \hat{\nabla}_i \chi_+ = 0$, then χ_+ satisfies the Killing spinor equations. For convenience, we also introduce an operator representing a general linear combination of the algebraic conditions,

$$\mathbb{A}^{(+,q_1,q_2)} = -q_1 A^{-1}\Gamma_z \mathbb{B}^{(+)} + q_2 \mathcal{A}, \quad (\text{B.7.28})$$

so that $\hat{\nabla}_i^{(+,q_1,q_2)} = \nabla_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2)}$, and the modified Dirac condition is

$$\Gamma^i \hat{\nabla}_i^{(+,q_1,q_2)} \chi_+ = \left(\Gamma^i \nabla_i + \Gamma^i \Psi_i^{(+)} + (10 - n) \mathbb{A}^{(+,q_1,q_2)} \right) \chi_+ = 0. \quad (\text{B.7.29})$$

The Laplacian expands into two terms,

$$\nabla^2 \|\chi_+\|^2 = 2\|\nabla \chi_+\|^2 + 2\langle \chi_+, \nabla^2 \chi_+ \rangle. \quad (\text{B.7.30})$$

The first term is then

$$\begin{aligned} 2\|\nabla \chi_+\|^2 &= 2\left\| \hat{\nabla}^{(+,q_1,q_2)} \chi_+ \right\|^2 - 4\left\langle \chi_+, \left(\Psi^{(+)\dagger} + \mathbb{A}^{(+,q_1,q_2)\dagger} \Gamma^i \right) \nabla_i \chi_+ \right\rangle \\ &\quad - 2\left\langle \chi_+, \left(\Psi^{(+)\dagger} + \mathbb{A}^{(+,q_1,q_2)\dagger} \Gamma^i \right) \left(\Psi_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2)} \right) \chi_+ \right\rangle \\ &= 2\left\| \hat{\nabla}^{(+,q_1,q_2)} \chi_+ \right\|^2 - 4\left\langle \chi_+, \Psi^{(+)\dagger} \nabla_i \chi_+ \right\rangle \\ &\quad - 2\left\langle \chi_+, \left(\Psi^{(+)\dagger} - \mathbb{A}^{(+,q_1,q_2)\dagger} \Gamma^i \right) \left(\Psi_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2)} \right) \chi_+ \right\rangle, \end{aligned}$$

while the second term is

$$\begin{aligned}
2\langle\chi_+, \nabla^2\chi_+\rangle &= 2\langle\chi_+, \Gamma^i\nabla_i(\Gamma^j\nabla_j\chi_+)\rangle + \frac{1}{2}R^{(10-n)}\|\chi_+\|^2 \\
&= \frac{1}{2}R^{(10-n)}\|\chi\|^2 - 2\langle\chi_+, \nabla_i\left(\Gamma^i\Gamma^j\Psi_j^{(+)} + (10-n)\Gamma^i\mathbb{A}^{(+,q_1,q_2)}\right)\chi_+\rangle \\
&\quad - 2\langle\chi_+, \left(\Gamma^i\Gamma^j\Psi_j^{(+)} + (10-n)\Gamma^i\mathbb{A}^{(+,q_1,q_2)}\right)\nabla_i\chi_+\rangle.
\end{aligned} \tag{B.7.31}$$

Thus, the full expansion is

$$\begin{aligned}
\nabla^2\|\chi_+\|^2 &= 2\left\|\hat{\nabla}^{(+,q_1,q_2)}\chi_+\right\|^2 + \frac{1}{2}R^{(10-n)}\|\chi_+\|^2 \\
&\quad + \langle\chi_+, \left[-4\Psi^{(+i)\dagger} - 2\Gamma^i\Gamma^j\Psi_j^{(+)} - 2(10-n)q_1A^{-1}\Gamma^{zi}\mathbb{B}^{(+)}\right. \\
&\quad \quad \left.- 2(10-n)q_2\Gamma^i\mathcal{A}\right]\nabla_i\chi_+\rangle \\
&\quad + \langle\chi_+, -2\left(\Psi^{(+i)\dagger} - \mathbb{A}^{(+,q_1,q_2)\dagger}\right)\left(\Psi_i^{(+)} + \Gamma_i\mathbb{A}^{(+,q_1,q_2)}\right)\chi_+\rangle \\
&\quad + \langle\chi_+, \nabla_i\left[-2\Gamma^i\Gamma^j\Psi_j^{(+)} - 2(10-n)q_1A^{-1}\Gamma^{zi}\mathbb{B}^{(+)}\right. \\
&\quad \quad \left.- 2(10-n)q_2\Gamma^i\mathcal{A}\right]\chi_+\rangle
\end{aligned} \tag{B.7.32}$$

where

$$\Psi_i^{(+)\dagger} = \frac{1}{2}A^{-1}\partial_iA - \frac{1}{8}\not{H}_i\Gamma_{11} + \frac{1}{8}S\Gamma_i + \frac{1}{16}\Gamma_i\not{H}\Gamma_{11} + \frac{1}{192}\Gamma_i\mathcal{G} \tag{B.7.33}$$

$$\mathbb{B}^{(+)\dagger} = -\frac{c}{2\ell} - \frac{1}{2}\not{\partial}A\Gamma_z - \frac{1}{8}AS\Gamma_z - \frac{1}{16}A\not{H}\Gamma_z\Gamma_{11} - \frac{1}{192}A\mathcal{G}\Gamma_z \tag{B.7.34}$$

$$\mathcal{A}^\dagger = \not{\partial}\Phi + \frac{5}{4}S + \frac{1}{12}\not{H}\Gamma_{11} - \frac{3}{8}\not{H}\Gamma_{11} + \frac{1}{96}\mathcal{G} \tag{B.7.35}$$

We would like to write the third term in the form $\alpha^i\nabla_i\|\chi_+\|^2 + \langle\chi_+, \mathcal{F}\Gamma^i\tilde{\nabla}_i\chi_+\rangle$. Expanded, this term is

$$\begin{aligned}
&\langle\chi_+, \left[-4\Psi^{(+i)\dagger} - 2\Gamma^i\Gamma^j\Psi_j^{(+)} - 2(10-n)q_1A^{-1}\Gamma^{zi}\mathbb{B}^{(+)} - 2(10-n)q_2\Gamma^i\mathcal{A}\right]\nabla_i\chi_+\rangle \\
&= \langle\chi_+, \left[\frac{(10-n)q_1c}{\ell}A^{-1}\Gamma^{zi} - [3 + (10-n)q_1]A^{-1}\partial^iA - [1 + (10-n)q_1]A^{-1}(\Gamma\not{\partial}A)^i\right. \\
&\quad - 2(10-n)q_2\Gamma^i\not{\partial}\Phi - \frac{12-n+(10-n)q_1+10(10-n)q_2}{4}S\Gamma^i \\
&\quad - \frac{1+2(10-n)q_2}{4}\not{H}^i\Gamma_{11} - \frac{3+2(10-n)q_2}{12}\Gamma\not{H}^i\Gamma_{11} \\
&\quad - \frac{(8-n)+(10-n)q_1+6(10-n)q_2}{8}\Gamma^i\not{H}\Gamma_{11} \\
&\quad \left. + \frac{(n-4)-(10-n)q_1-2(10-n)q_2}{96}\Gamma^i\mathcal{G}\right]\nabla_i\chi_+\rangle.
\end{aligned} \tag{B.7.36}$$

This fixes $q_2 = -\frac{1}{10-n}$ and $q_1 = \frac{n-2}{10-n}$. The term is thus

$$\begin{aligned} & \left\langle \chi_+, \left[-4\Psi^{(+)\dagger} - 2\Gamma^i\Gamma^j\Psi_j^{(+)} - 2(n-2)A^{-1}\Gamma^{zi}\mathbb{B}^{(+)} + 2\Gamma^i\mathcal{A} \right] \nabla_i \chi_+ \right\rangle \\ &= \left\langle \chi_+, \left[\frac{(n-2)c}{\ell} A^{-1}\Gamma_{zi} - (n+1)A^{-1}\partial_i A - (n-1)A^{-1}(\Gamma\phi A)_i \right. \right. \\ & \quad \left. \left. + 2\Gamma_i\phi\Phi - \frac{1}{12}(\Gamma H)_i\Gamma_{11} + \frac{1}{4}\#_i\Gamma_{11} \right] \nabla^i \chi_+ \right\rangle, \end{aligned} \quad (\text{B.7.37})$$

so

$$\mathcal{F} = \frac{(n-2)c}{\ell} A^{-1}\Gamma_z + (n-1)A^{-1}\phi A - 2\phi\Phi - \frac{1}{12}\#\Gamma_{11} \quad (\text{B.7.38})$$

and $\alpha_i = -nA^{-1}\partial^i A + 2\partial^i\Phi$.

Combining this with the fourth term in (B.8.6), we find

$$\begin{aligned}
& \left\langle \chi_+, -2 \left(\Psi^{(+)\dagger} + \frac{n-2}{10-n} A^{-1} \mathbb{B}^{(+)\dagger} \Gamma^{zi} + \frac{1}{10-n} \mathcal{A}^\dagger \Gamma^i + \frac{1}{2} \mathcal{F} \Gamma^i \right) \right. \\
& \quad \times \left. \left(\Psi_i^{(+)} + \frac{n-2}{10-n} A^{-1} \Gamma_{zi} \mathbb{B}^{(+)} - \frac{1}{10-n} \Gamma_i \mathcal{A} \right) \chi_+ \right\rangle \\
&= \left\langle \chi_+, -2 \left[\frac{(9-n)(n-2)c}{2(10-n)\ell} A^{-1} \Gamma^{zi} - \frac{n^2-9n-2}{2(10-n)} A^{-1} \partial^i A + \frac{n^2-10n+8}{2(10-n)} A^{-1} \Gamma \not\partial^i A \right. \right. \\
& \quad - \frac{9-n}{10-n} \not\partial \Phi \Gamma^i + \frac{11-n}{4(10-n)} S \Gamma^i - \frac{1}{4(10-n)} \not{H}^i \Gamma_{11} - \frac{8-n}{24(10-n)} \Gamma \not{H}^i \Gamma_{11} \\
& \quad + \frac{9-n}{8(10-n)} \Gamma \not{F}^i \Gamma_{11} + \frac{1}{4(10-n)} \not{F}^i \Gamma_{11} + \frac{7-n}{96(10-n)} \Gamma \not{G}^i + \frac{1}{8(10-n)} \not{G}^i \left. \right] \\
& \quad \times \left[-\frac{(n-2)c}{2(10-n)\ell} A^{-1} \Gamma_{zi} + \frac{4}{10-n} A^{-1} \partial_i A + \frac{n-2}{2(10-n)} A^{-1} \Gamma \not\partial_i A - \frac{1}{10-n} \Gamma_i \not\partial \Phi \right. \\
& \quad - \frac{1}{4(10-n)} S \Gamma_i + \frac{8-n}{8(10-n)} \not{H}_i \Gamma_{11} - \frac{1}{12(10-n)} \Gamma \not{H}_i \Gamma_{11} \\
& \quad + \frac{1}{8(10-n)} \Gamma \not{F}_i \Gamma_{11} - \frac{9-n}{4(10-n)} \not{F}_i \Gamma_{11} + \frac{1}{32(10-n)} \Gamma \not{G}_i - \frac{7-n}{24(10-n)} \not{G}_i \left. \right] \chi_+ \right\rangle \tag{B.7.39}
\end{aligned}$$

$$\begin{aligned}
&= \left\langle \chi_+, \left[-\frac{(9-n)(n-2)^2}{2(10-n)\ell^2} A^{-2} + \frac{n^3-11n^2+18n-16}{2(10-n)} A^{-2} (dA)^2 \right. \right. \\
& \quad - \frac{2(n^2-10n+8)}{10-n} A^{-1} \partial^i A \partial_i \Phi - \frac{2(9-n)}{10-n} (d\Phi)^2 - \frac{(n-2)c}{2(10-n)\ell} A^{-1} S \Gamma_z \\
& \quad - \frac{n-2}{2(10-n)} A^{-1} \not\partial A S + \frac{1}{10-n} S \not\partial \Phi + \frac{11-n}{8(10-n)} S^2 + \frac{(n-2)c}{6(10-n)\ell} A^{-1} \not{H} \Gamma_z \Gamma_{11} \\
& \quad - \frac{n-2}{6(10-n)} A^{-1} \partial_i A \Gamma \not{H}^i \Gamma_{11} + \frac{1}{3(10-n)} \partial_i \Phi \Gamma \not{H}^i \Gamma_{11} - \frac{8-n}{24(10-n)} S \not{H} \Gamma_{11} \\
& \quad - \frac{1}{72(10-n)} \not{H} \not{H} - \frac{1}{24} H^2 + \frac{(n-2)c}{4(10-n)\ell} A^{-1} \not{F} \Gamma_z \Gamma_{11} + \frac{n-2}{4(10-n)} A^{-1} \partial_i A \Gamma \not{F}^i \Gamma_{11} \\
& \quad - \frac{1}{2(10-n)} \partial_i \Phi \Gamma \not{F}^i \Gamma_{11} - \frac{12-n}{48(10-n)} \not{H} \not{F} + \frac{1}{8} \not{H}^i \not{F}^i - \frac{11-n}{32(10-n)} \not{F} \not{F} \\
& \quad - \frac{1}{8} F^2 + \frac{(n-2)c}{16(10-n)\ell} A^{-1} \not{G} \Gamma_z + \frac{n-2}{16(10-n)} A^{-1} \partial_i A \Gamma \not{G}^i \\
& \quad - \frac{1}{8(10-n)} \partial_i \Phi \Gamma \not{G}^i + \frac{7-n}{96(10-n)} S \not{G} - \frac{n-4}{576(10-n)} \not{H} \not{G} \Gamma_{11} \\
& \quad \left. - \frac{1}{64(10-n)} \not{F} \not{G} \Gamma_{11} + \frac{n-1}{48 \cdot 96(10-n)} \not{G} \not{G} - \frac{1}{96} G^2 \right] \chi_+ \right\rangle \tag{B.7.40}
\end{aligned}$$

Using the field equations and Bianchi identities, the last term is

$$\begin{aligned} & \left\langle \chi_+, \nabla_i \left[-2\Gamma^i \Gamma^j \Psi_j^{(+)} - 2(n-2)A^{-1}\Gamma^{zi}\mathbb{B}^{(+)} + 2\Gamma^i \mathcal{A} \right] \chi_+ \right\rangle \\ &= \left\langle \chi_+, \left[-(n-1)\nabla^2 \ln A + 2\nabla^2 \Phi + \frac{1}{2}\not\partial S - \frac{1}{48}d\sharp\Gamma_{11} + \frac{1}{12}d\sharp\Gamma_{11} + \frac{1}{240}d\mathcal{G} \right] \chi_+ \right\rangle \quad (\text{B.7.41}) \end{aligned}$$

$$\begin{aligned} &= \left\langle \chi_+, \left[\frac{(n-1)^2}{\ell^2}A^{-2} + (n-1)nA^{-2}(dA)^2 - 2(2n-1)A^{-1}\partial^i A \partial_i \Phi + 4(d\Phi)^2 \right. \right. \\ &\quad \left. \left. + \frac{1}{2}S\not\partial\Phi + \frac{11-n}{4}S^2 + \frac{1}{12}S\sharp\Gamma_{11} - \frac{1}{6}H^2 + \frac{1}{4}\partial_i\Phi\Gamma\sharp^i\Gamma_{11} \right. \right. \\ &\quad \left. \left. + \frac{1}{24}\sharp\sharp - \frac{1}{8}\sharp^i\sharp_i + \frac{7-n}{8}F^2 + \frac{1}{48}\partial_i\Phi\Gamma\mathcal{G}^i \right] \chi_+ \right\rangle \quad (\text{B.7.42}) \end{aligned}$$

while the curvature term is

$$\begin{aligned} \frac{1}{2}R^{(10-n)}\|\chi_+\|^2 &= \left\langle \chi_+, \left[-\frac{(n-1)n}{2\ell^2}A^{-2} - \frac{(n-1)n}{2}A^{-2}(dA)^2 + 2nA^{-1}\partial^i A \partial_i \Phi - 2(d\Phi)^2 \right. \right. \\ &\quad \left. \left. - \frac{10-n}{4}S^2 + \frac{5}{24}H^2 + \frac{n-6}{8}F^2 + \frac{n-2}{96}G^2 \right] \chi_+ \right\rangle. \quad (\text{B.7.43}) \end{aligned}$$

The sum of (B.7.40), (B.7.42), and (B.7.43), i.e. the second through fifth terms of (B.8.6), is

$$\begin{aligned} & \left\langle \chi_+, \left[\frac{4(n-2)}{(10-n)\ell^2}A^{-2} + \frac{4(n-2)}{10-n}A^{-2}(dA)^2 - \frac{2(n-2)}{10-n}A^{-1}\partial^i A \partial_i \Phi + \frac{2}{10-n}(d\Phi)^2 \right. \right. \\ & \quad - \frac{(n-2)c}{2(10-n)\ell}A^{-1}S\Gamma_z - \frac{n-2}{2(10-n)}A^{-1}\not\partial AS + \frac{12-n}{2(10-n)}S\not\partial\Phi + \frac{31-3n}{8(10-n)}S^2 \\ & \quad + \frac{(n-2)c}{6(10-n)\ell}A^{-1}\sharp\Gamma_z\Gamma_{11} - \frac{n-2}{6(10-n)}A^{-1}\partial_i A \Gamma\sharp^i\Gamma_{11} + \frac{1}{3(10-n)}\partial_i\Phi\Gamma\sharp^i\Gamma_{11} \\ & \quad + \frac{12-n}{24(10-n)}S\sharp\Gamma_{11} - \frac{1}{72(10-n)}\sharp\sharp + \frac{(n-2)c}{4(10-n)\ell}A^{-1}\sharp\Gamma_z\Gamma_{11} \\ & \quad + \frac{n-2}{4(10-n)}A^{-1}\partial_i A \Gamma\sharp^i\Gamma_{11} + \frac{8-n}{4(10-n)}\partial_i\Phi\Gamma\sharp^i\Gamma_{11} + \frac{8-n}{48(10-n)}\sharp\sharp \\ & \quad - \frac{11-n}{32(10-n)}\sharp\sharp + \frac{(n-2)c}{16(10-n)\ell}A^{-1}\mathcal{G}\Gamma_z + \frac{n-2}{16(10-n)}A^{-1}\partial_i A \Gamma\mathcal{G}^i \\ & \quad - \frac{n-4}{48(10-n)}\partial_i\Phi\Gamma\mathcal{G}^i + \frac{7-n}{96(10-n)}S\mathcal{G} - \frac{n-4}{576(10-n)}\sharp\mathcal{G}\Gamma_{11} \\ & \quad \left. - \frac{1}{64(10-n)}\sharp\mathcal{G}\Gamma_{11} + \frac{n-1}{48 \cdot 96(10-n)}\mathcal{G}\mathcal{G} \right] \chi_+ \right\rangle \quad (\text{B.7.44}) \end{aligned}$$

Comparing this to

$$\begin{aligned} \left\| \mathbb{B}^{(+)} \chi_+ \right\|^2 = & \left\langle \chi_+, \left[\frac{1}{4\ell^2} + \frac{1}{4}(dA)^2 + \frac{c}{8\ell} A S \Gamma_z + \frac{1}{8} A \not{D} A S + \frac{1}{64} A^2 S^2 \right. \right. \\ & + \frac{c}{16\ell} A \not{F} \Gamma_z \Gamma_{11} + \frac{1}{16} A \partial_i A (\Gamma/F)^i - \frac{1}{256} A^2 \not{F} \not{F} \\ & + \frac{c}{192\ell} A \not{G} \Gamma_z + \frac{1}{192} A \partial_i A \Gamma \not{G}^i + \frac{1}{768} A^2 S \not{G} \\ & \left. \left. - \frac{1}{16 \cdot 96} \not{F} \not{G} \Gamma_{11} + \frac{1}{192^2} A^2 \not{G} \not{G} \right] \chi_+ \right\rangle \end{aligned} \quad (\text{B.7.45})$$

$$\begin{aligned} \left\langle \Gamma_z \mathbb{B}^{(+)} \chi_+, \mathcal{A} \chi_+ \right\rangle = & \left\langle \chi_+, \left[-\frac{1}{2} \partial^i A \partial_i \Phi + \frac{c}{24\ell} \not{H} \Gamma_z \Gamma_{11} - \frac{5c}{8\ell} S \Gamma_z - \frac{5}{8} \not{D} A S - \frac{1}{8} A S \not{D} \Phi \right. \right. \\ & - \frac{5}{32} A S^2 - \frac{1}{24} \partial_i A (\Gamma/H)^i \Gamma_{11} - \frac{1}{96} A S \not{H} \Gamma_{11} \\ & - \frac{3c}{16\ell} \not{F} \Gamma_z \Gamma_{11} - \frac{3}{16} \partial_i A (\Gamma/F)^i \Gamma_{11} - \frac{1}{16} A \partial_i \Phi (\Gamma/F)^i \Gamma_{11} \\ & - \frac{A}{192} \not{H} \not{F} + \frac{3}{128} A \not{F} \not{F} - \frac{c}{192\ell} \not{G} \Gamma_z - \frac{1}{192} \partial_i A \Gamma \not{G}^i \\ & - \frac{1}{192} A \partial_i \Phi \Gamma \not{G}^i - \frac{1}{128} A S \not{G} - \frac{1}{24 \cdot 96} A \not{H} \not{G} \Gamma_{11} \\ & \left. \left. + \frac{1}{384} A \not{F} \not{G} \Gamma_{11} - \frac{1}{96 \cdot 192} A \not{G} \not{G} \right] \chi_+ \right\rangle \end{aligned} \quad (\text{B.7.46})$$

$$\begin{aligned} \left\| \mathcal{A} \chi_+ \right\|^2 = & \left\langle \chi_+, \left[(d\Phi)^2 + \frac{1}{6} \partial_i \Phi (\Gamma/H)^i \Gamma_{11} - \frac{1}{144} \not{H} \not{H} + \frac{5}{2} S \not{D} \Phi + \frac{5}{24} S \not{H} \Gamma_{11} \right. \right. \\ & + \frac{25}{16} S^2 + \frac{3}{4} \partial_i \Phi (\Gamma/F)^i \Gamma_{11} + \frac{1}{16} \not{H} \not{F} - \frac{9}{64} \not{F} \not{F} \\ & \left. \left. + \frac{1}{48} \partial_i \Phi \Gamma \not{G}^i + \frac{5}{192} S \not{G} + \frac{1}{576} \not{H} \not{G} \Gamma_{11} - \frac{1}{128} \not{F} \not{G} \Gamma_{11} + \frac{1}{96^2} \not{G} \not{G} \right] \chi_+ \right\rangle \end{aligned} \quad (\text{B.7.47})$$

we find that

$$\begin{aligned} & \nabla^2 \left\| \chi_+ \right\|^2 + (n A^{-1} \partial_i A - 2 \partial_i \Phi) \nabla^i \left\| \chi_+ \right\|^2 \\ & = \left\| \hat{\nabla} \chi_+ \right\|^2 + \frac{16(n-2)}{10-n} A^{-2} \left\| \mathbb{B}^{(+)} \chi_+ \right\|^2 + \frac{4(n-2)}{10-n} A^{-1} \left\langle \Gamma_z \mathbb{B}^{(+)} \chi_+, \mathcal{A} \chi_+ \right\rangle \\ & \quad + \frac{2}{10-n} \left\| \mathcal{A} \chi_+ \right\|^2. \end{aligned} \quad (\text{B.7.48})$$

B.8 Common Sector AdS_n, $n \geq 4$

We introduce a new operator,

$$\hat{\nabla}_i^{(+, q_1, q_2)} = \nabla_i^{(+)} + q_1 A^{-1} \Gamma_{zi} \mathbb{B}^{(+)} + q_2 \Gamma_i \mathcal{A}, \quad (\text{B.8.1})$$

with the intention to demonstrate that, for an appropriately chosen value of q_1 and q_2 , if $\Gamma^i \hat{\nabla}_i \chi_+ = 0$, then χ_+ satisfies the Killing spinor equations. For convenience, we also intro-

duce an operator representing a general linear combination of the algebraic conditions,

$$\mathbb{A}^{(+,q_1,q_2,\kappa)} = -q_1 e^{\kappa\Phi} A^{-1} \Gamma_z \mathbb{B}^{(+)} + q_2 \mathcal{A}, \quad (\text{B.8.2})$$

so that $\hat{\nabla}_i^{(+,q_1,q_2,\kappa)} = \nabla_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2,\kappa)}$, and the modified Dirac condition is

$$\Gamma^i \hat{\nabla}_i^{(+,q_1,q_2,\kappa)} \chi_+ = \left(\Gamma^i \nabla_i + \Gamma^i \Psi_i^{(+)} + (10-n) \mathbb{A}^{(+,q_1,q_2,\kappa)} \right) \chi_+ = 0. \quad (\text{B.8.3})$$

The Laplacian expands into two terms,

$$\nabla^2 \|\chi_+\|^2 = 2 \|\nabla \chi_+\|^2 + 2 \langle \chi_+, \nabla^2 \chi_+ \rangle. \quad (\text{B.8.4})$$

The first term is then

$$\begin{aligned} 2 \|\nabla \chi_+\|^2 &= 2 \left\| \hat{\nabla}^{(+,q_1,q_2,\kappa)} \chi_+ \right\|^2 - 4 \left\langle \chi_+, \left(\Psi^{(+i)\dagger} + \mathbb{A}^{(+,q_1,q_2,\kappa)\dagger} \Gamma^i \right) \nabla_i \chi_+ \right\rangle \\ &\quad - 2 \left\langle \chi_+, \left(\Psi^{(+i)\dagger} + \mathbb{A}^{(+,q_1,q_2,\kappa)\dagger} \Gamma^i \right) \left(\Psi_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2,\kappa)} \right) \chi_+ \right\rangle \\ &= 2 \left\| \hat{\nabla}^{(+,q_1,q_2,\kappa)} \chi_+ \right\|^2 - 4 \left\langle \chi_+, \Psi^{(+i)\dagger} \nabla_i \chi_+ \right\rangle \\ &\quad - 2 \left\langle \chi_+, \left(\Psi^{(+i)\dagger} - \mathbb{A}^{(+,q_1,q_2,\kappa)\dagger} \Gamma^i \right) \left(\Psi_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2,\kappa)} \right) \chi_+ \right\rangle, \end{aligned}$$

while the second term is

$$\begin{aligned} 2 \langle \chi_+, \nabla^2 \chi_+ \rangle &= 2 \langle \chi_+, \Gamma^i \nabla_i (\Gamma^j \nabla_j \chi_+) \rangle + \frac{1}{2} R^{(10-n)} \|\chi_+\|^2 \\ &= \frac{1}{2} R^{(10-n)} \|\chi\|^2 - 2 \left\langle \chi_+, \nabla_i \left(\Gamma^i \Gamma^j \Psi_j^{(+)} + (10-n) \Gamma^i \mathbb{A}^{(+,q_1,q_2,\kappa)} \right) \chi_+ \right\rangle \\ &\quad - 2 \left\langle \chi_+, \left(\Gamma^i \Gamma^j \Psi_j^{(+)} + (10-n) \Gamma^i \mathbb{A}^{(+,q_1,q_2,\kappa)} \right) \nabla_i \chi_+ \right\rangle. \end{aligned} \quad (\text{B.8.5})$$

Thus, the full expansion is

$$\begin{aligned} \nabla^2 \|\chi_+\|^2 &= 2 \left\| \hat{\nabla}^{(+,q_1,q_2,\kappa)} \chi_+ \right\|^2 + \frac{1}{2} R^{(10-n)} \|\chi_+\|^2 \\ &\quad + \left\langle \chi_+, \left[-4 \Psi^{(+i)\dagger} - 2 \Gamma^i \Gamma^j \Psi_j^{(+)} - 2(10-n) q_1 e^{\kappa\Phi} A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} - 2(10-n) q_2 \Gamma^i \mathcal{A} \right] \nabla_i \chi_+ \right\rangle \\ &\quad + \left\langle \chi_+, -2 \left(\Psi^{(+i)\dagger} - \mathbb{A}^{(+,q_1,q_2,\kappa)\dagger} \Gamma^i \right) \left(\Psi_i^{(+)} + \Gamma_i \mathbb{A}^{(+,q_1,q_2,\kappa)} \right) \chi_+ \right\rangle \\ &\quad + \left\langle \chi_+, \nabla_i \left[-2 \Gamma^i \Gamma^j \Psi_j^{(+)} - 2(10-n) q_1 e^{\kappa\Phi} A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} - 2(10-n) q_2 \Gamma^i \mathcal{A} \right] \chi_+ \right\rangle \end{aligned} \quad (\text{B.8.6})$$

The field equations and Bianchi identity are

$$dH = 0 \quad (\text{B.8.7})$$

$$\nabla^2 \Phi = -n A^{-1} \partial^i A \partial_i \Phi + 2(d\Phi)^2 - \frac{1}{12} H^2 \quad (\text{B.8.8})$$

$$\nabla^k H_{ijk} = -n A^{-1} \partial^k A H_{ijk} + 2 \partial^k \Phi H_{ijk} \quad (\text{B.8.9})$$

and the Einstein equation separates into an AdS component,

$$\nabla^2 \ln A = -(n-1) \ell^{-2} A^{-2} - n A^{-2} (dA)^2, \quad (\text{B.8.10})$$

and a transverse component, which contracts to

$$R^{(10-n)} = n\nabla^2 \ln A + nA^{-2}(dA)^2 + \frac{1}{4}H^2 - 2\nabla^2\Phi \quad (\text{B.8.11})$$

$$= -\frac{n(n-1)}{\ell^2}A^{-2} - n(n-1)A^{-2}(dA)^2 + \frac{5}{12}H^2 + 2nA^{-1}\partial_i A \partial^i \Phi - 4(d\Phi)^2. \quad (\text{B.8.12})$$

$$\Psi_i^{(+)} = \frac{1}{2}A^{-1}\partial_i A + \frac{1}{8}\not{H}_i \Gamma_{11} \quad (\text{B.8.13})$$

$$\mathbb{B}^{(+)} = -\frac{c}{2\ell} + \frac{1}{2}\not{\partial} A \Gamma_z \quad (\text{B.8.14})$$

$$\mathcal{A} = \not{\partial}\Phi + \frac{1}{12}\not{H} \Gamma_{11} \quad (\text{B.8.15})$$

$$\Psi_i^{(+)\dagger} = \frac{1}{2}A^{-1}\partial_i A - \frac{1}{8}\not{H}_i \Gamma_{11} \quad (\text{B.8.16})$$

$$\mathbb{B}^{(+)\dagger} = -\frac{c}{2\ell} - \frac{1}{2}\not{\partial} A \Gamma_z \quad (\text{B.8.17})$$

$$\mathcal{A}^\dagger = \not{\partial}\Phi + \frac{1}{12}\not{H} \Gamma_{11} \quad (\text{B.8.18})$$

$$\begin{aligned} & \left\langle \chi_+, \left[-4\Psi^{(+)\dagger} - 2\Gamma^i \Gamma^j \Psi_j^{(+)} - 2(10-n)q_1 e^{\kappa\Phi} A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} - 2(10-n)q_2 \Gamma^i \mathcal{A} \right] \nabla_i \chi_+ \right\rangle \\ &= \left\langle \chi_+, \left[\frac{(10-n)q_1 c}{\ell} e^{\kappa\Phi} A^{-1} \Gamma^{zi} - [3 + (10-n)q_1 e^{\kappa\Phi}] A^{-1} \partial^i A \right. \right. \\ & \quad - [1 + (10-n)q_1 e^{\kappa\Phi}] A^{-1} (\Gamma \not{\partial} A)^i - 2(10-n)q_2 \Gamma^i \not{\partial}\Phi \\ & \quad \left. \left. - \frac{1 + 2(10-n)q_2}{4} \not{H}^i \Gamma_{11} - \frac{3 + 2(10-n)q_2}{12} \Gamma \not{H}^i \Gamma_{11} \right] \nabla_i \chi_+ \right\rangle. \end{aligned} \quad (\text{B.8.19})$$

This fixes $q_2 = -\frac{1}{10-n}$.

$$\mathcal{F} = \frac{(10-n)q_1 c}{\ell} e^{\kappa\Phi} A^{-1} \Gamma_z + [1 + (10-n)q_1 e^{\kappa\Phi}] A^{-1} \not{\partial} A - 2\not{\partial}\Phi - \frac{1}{12}\not{H} \Gamma_{11} \quad (\text{B.8.20})$$

$$\alpha^i = -[2 + (10-n)q_1 e^{\kappa\Phi}] A^{-1} \partial^i A + 2\partial^i \Phi \quad (\text{B.8.21})$$

$$\Psi^{(+)\dagger} + q_1 e^{\kappa\Phi} A^{-1} \mathbb{B}^{(+)\dagger} \Gamma^{zi} + \frac{1}{10-n} \mathcal{A}^\dagger \Gamma^i + \frac{1}{2} \mathcal{F} \Gamma^i \quad (\text{B.8.22})$$

$$\begin{aligned} &= \frac{(9-n)q_1 c}{2\ell} e^{\kappa\Phi} A^{-1} \Gamma^{zi} + \frac{2 + (9-n)q_1 e^{\kappa\Phi}}{2} A^{-1} \partial^i A - \frac{1 + (9-n)q_1 e^{\kappa\Phi}}{2} A^{-1} \Gamma \not\partial^i A \\ &\quad - \frac{9-n}{10-n} \not\partial \Phi \Gamma^i - \frac{1}{4(10-n)} \not{H}^i \Gamma_{11} - \frac{8-n}{24(10-n)} \Gamma \not{H}^i \Gamma_{11} \\ \Psi_i^{(+)} + q_1 e^{\kappa\Phi} A^{-1} \Gamma_{zi} \mathbb{B}^{(+)} - \frac{1}{10-n} \Gamma_i \mathcal{A} \quad (\text{B.8.23}) \end{aligned}$$

$$\begin{aligned} &= -\frac{q_1 c}{2\ell} e^{\kappa\Phi} A^{-1} \Gamma_{zi} + \frac{1 + q_1 e^{\kappa\Phi}}{2} A^{-1} \partial_i A + \frac{q_1}{2} e^{\kappa\Phi} A^{-1} \Gamma \not\partial_i A - \frac{1}{10-n} \Gamma_i \not\partial \Phi \\ &\quad + \frac{8-n}{8(10-n)} \not{H}_i \Gamma_{11} - \frac{1}{12(10-n)} \Gamma \not{H}_i \Gamma_{11} \end{aligned}$$

$$\begin{aligned} &\left\langle \chi_+, -2 \left(\Psi^{(+)\dagger} + q_1 e^{\kappa\Phi} A^{-1} \mathbb{B}^{(+)\dagger} \Gamma^{zi} + \frac{1}{10-n} \mathcal{A}^\dagger \Gamma^i + \frac{1}{2} \mathcal{F} \Gamma^i \right) \right. \\ &\quad \left. \times \left(\Psi_i^{(+)} + q_1 e^{\kappa\Phi} A^{-1} \Gamma_{zi} \mathbb{B}^{(+)} - \frac{1}{10-n} \Gamma_i \mathcal{A} \right) \chi_+ \right\rangle \\ &= \left\langle \chi_+, \left[-\frac{(10-n)(9-n)q_1^2}{2} e^{2\kappa\Phi} A^{-2} \left(\frac{1}{\ell^2} + (dA)^2 \right) - [1 + (10-n)q_1 e^{\kappa\Phi}] A^{-2} (dA)^2 \right. \right. \\ &\quad + 2[1 + (9-n)q_1 e^{\kappa\Phi}] A^{-1} \partial_i A \partial^i \Phi - \frac{2(9-n)}{10-n} (d\Phi)^2 + \frac{q_1 c}{6\ell} e^{\kappa\Phi} A^{-1} \not{H} \Gamma_z \Gamma_{11} \\ &\quad \left. \left. - \frac{q_1}{6} e^{\kappa\Phi} A^{-1} \partial_i A \Gamma \not{H}^i \Gamma_{11} + \frac{1}{3(10-n)} \partial_i \Phi \Gamma \not{H}^i \Gamma_{11} - \frac{1}{72(10-n)} \not{H} \not{H} - \frac{1}{24} H^2 \right] \chi_+ \right\rangle \quad (\text{B.8.24}) \end{aligned}$$

$$\begin{aligned} &\left\langle \chi_+, \nabla_i \left[-2\Gamma^i \Gamma^j \Psi_j^{(+)} - 2(10-n)q_1 e^{\kappa\Phi} A^{-1} \Gamma^{zi} \mathbb{B}^{(+)} + 2\Gamma^i \mathcal{A} \right] \chi_+ \right\rangle \\ &= \left\langle \chi_+, \left[-[1 + (10-n)q_1 e^{\kappa\Phi}] \nabla^2 \ln A - (10-n)\kappa q_1 e^{\kappa\Phi} A^{-1} \partial_i A \partial^i \Phi + 2\nabla^2 \Phi - \frac{1}{48} d\mathcal{H} \Gamma_{11} \right] \chi_+ \right\rangle \quad (\text{B.8.25}) \end{aligned}$$

$$\begin{aligned} &= \left\langle \chi_+, \left[(n-1)[1 + (10-n)q_1 e^{\kappa\Phi}] A^{-2} \left(\frac{1}{\ell^2} + (dA)^2 \right) + [1 + (10-n)q_1 e^{\kappa\Phi}] A^{-2} (dA)^2 \right. \right. \\ &\quad \left. \left. - [2n + (10-n)\kappa q_1 e^{\kappa\Phi}] A^{-1} \partial_i A \partial^i \Phi + 4(d\Phi)^2 - \frac{1}{6} H^2 \right] \chi_+ \right\rangle \quad (\text{B.8.26}) \end{aligned}$$

$$\frac{1}{2} R^{(10-n)} \|\chi_+\|^2 = \left\langle \chi_+, \left[-\frac{(n-1)n}{2} A^{-2} \left(\frac{1}{\ell^2} + (dA)^2 \right) + n A^{-1} \partial^i A \partial_i \Phi - 2(d\Phi)^2 + \frac{5}{24} H^2 \right] \chi_+ \right\rangle$$

The sum is

$$\begin{aligned} &\left\langle \chi_+, \left[-\frac{1}{2} p(n, q_1 e^{\kappa\Phi}) A^{-2} \left(\frac{1}{\ell^2} + (dA)^2 \right) + [(2-n) + 2(9-n)q_1 e^{\kappa\Phi} + (10-n)\kappa q_1 e^{\kappa\Phi}] A^{-1} \partial_i A \partial^i \Phi + \frac{2}{10-n} (d\Phi)^2 \right. \right. \\ &\quad \left. \left. + \frac{q_1 c}{6\ell} e^{\kappa\Phi} A^{-1} \not{H} \Gamma_z \Gamma_{11} - \frac{q_1}{6} e^{\kappa\Phi} A^{-1} \partial_i A \Gamma \not{H}^i \Gamma_{11} + \frac{1}{3(10-n)} \partial_i \Phi \Gamma \not{H}^i \Gamma_{11} - \frac{1}{72(10-n)} \not{H} \not{H} \right] \chi_+ \right\rangle \quad (\text{B.8.27}) \end{aligned}$$

where

$$p(n, q_1 e^{\kappa\Phi}) = [(n-1) - (10-n)q_1 e^{\kappa\Phi}]^2 - [(n-1) + (10-n)q_1^2 e^{2\kappa\Phi}] \quad (\text{B.8.28})$$

$$\begin{aligned} \|\mathbb{B}^{(+)}\chi_+\|^2 &= \left\langle \chi_+, \left[\frac{1}{4} \left(\frac{1}{\ell^2} + (dA)^2 \right) \right] \chi_+ \right\rangle \\ \langle \Gamma_z \mathbb{B}^{(+)}\chi_+, \mathcal{A}\chi_+ \rangle &= \left\langle \chi_+, \left[-\frac{1}{2} \partial^i A \partial_i \Phi + \frac{c}{24\ell} \# \Gamma_z \Gamma_{11} - \frac{1}{24} \partial_i A (\Gamma \#)^i \Gamma_{11} \right] \chi_+ \right\rangle \\ \|\mathcal{A}\chi_+\|^2 &= \left\langle \chi_+, \left[(d\Phi)^2 + \frac{1}{6} \partial_i \Phi (\Gamma \#)^i \Gamma_{11} - \frac{1}{144} \# \# \right] \chi_+ \right\rangle \end{aligned}$$

$$\begin{aligned} &\nabla^2 \|\chi_+\|^2 + ([2 + (10-n)q_1 e^{\kappa\Phi}] A^{-1} \partial^i A - 2\partial^i \Phi) \nabla^i \|\chi_+\|^2 \\ &= \|\hat{\nabla}\chi_+\|^2 - 2p(n, q_1 e^{\kappa\Phi}) A^{-2} \|\mathbb{B}^{(+)}\chi_+\|^2 + 4q_1 e^{\kappa\Phi} A^{-1} \langle \Gamma_z \mathbb{B}^{(+)}\chi_+, \mathcal{A}\chi_+ \rangle \\ &\quad + \frac{2}{10-n} \|\mathcal{A}\chi_+\|^2 \end{aligned} \quad (\text{B.8.29})$$

only if

$$(2-n) + 2(9-n)q_1 e^{\kappa\Phi} + (10-n)\kappa q_1 e^{\kappa\Phi} = -2q_1 e^{\kappa\Phi} \quad (\text{B.8.30})$$

rearranged, this is

$$q_1 e^{\kappa\Phi} = \frac{n-2}{(\kappa+2)(10-n)} \quad (\text{B.8.31})$$

which, if we choose $\kappa = -1$, implies that $q_1 e^{\kappa\Phi} = \frac{n-2}{10-n}$ and that the right side of the equation is positive definite.

B.9 IIB AdS-direction KSEs

B.9.1 Equations

The Killing spinor equations are

$$\begin{aligned} & \left(\nabla_M - \frac{i}{2} Q_M + \frac{i}{48} F_{MN_1N_2N_3N_4} \Gamma^{N_1N_2N_3N_4} \right) \epsilon \\ & - \frac{1}{96} (\Gamma_M{}^{N_1N_2N_3} G_{N_1N_2N_3} - 9 G_{MN_1N_2} \Gamma^{N_1N_2}) C * \epsilon = 0 \end{aligned} \quad (\text{B.9.1})$$

and

$$P_M \Gamma^M C * \epsilon + \frac{1}{24} G_{N_1N_2N_3} \Gamma^{N_1N_2N_3} \epsilon = 0, \quad (\text{B.9.2})$$

where $C = \Gamma^{6789}$ and $*$ is the complex conjugation operator. The metric is given by

$$ds^2 = A^2 ds^2(\text{AdS}_k) + ds^2(\mathcal{S}) \quad (\text{B.9.3})$$

$$= \frac{A^2}{z^2} (\eta_{\mu\nu} dx^\mu dx^\nu + dz^2) + (g_S)_{ij} dy^i dy^j \quad (\text{B.9.4})$$

and the frame forms are

$$e^\mu = \frac{A}{z} dx^\mu \quad (\text{B.9.5})$$

$$e^{\underline{z}} = \frac{A}{z} dz \quad (\text{B.9.6})$$

$$e^{\underline{i}} = e^{\underline{i}}_{\mathcal{S}} \quad (\text{B.9.7})$$

where indices are underlined to indicate that they are frame indices. The non-zero components of the spin connection are thus

$$\Omega_{\mu, \underline{\nu} \underline{z}} = -\frac{1}{z} \eta_{\mu\nu} \quad (\text{B.9.8})$$

$$\Omega_{\mu, \underline{\nu} \underline{i}} = \frac{1}{z} \eta_{\mu\nu} \partial_{\underline{i}} A \quad (\text{B.9.9})$$

$$\Omega_{z, \underline{z} \underline{i}} = \frac{1}{z} \partial_{\underline{i}} A \quad (\text{B.9.10})$$

$$\Omega_{i, \underline{j} \underline{k}} = (\Omega_{\mathcal{S}})_{i, \underline{j} \underline{k}}. \quad (\text{B.9.11})$$

B.9.2 $k \geq 8$

For $k \geq 8$, $F = 0$ and $G = 0$, as the transverse space is not large enough to support either of these fields. The first Killing spinor equation, (B.9.1), in the $\mu = 0, \dots, k-2$ directions reduces to

$$\partial_\mu \epsilon - \frac{1}{2z} \Gamma_\mu{}^{\underline{z}} \epsilon + \frac{1}{2z} \partial_{\underline{i}} A \Gamma_\mu{}^{\underline{i}} \epsilon = 0, \quad (\text{B.9.12})$$

where we have used the spin connection from equations (B.9.8)–(B.9.11). This system of partial differential equations only if the integrability condition, $\partial_\mu \partial_\nu \epsilon = \partial_\nu \partial_\mu \epsilon$ is met. This condition

can be rewritten as

$$[\not{F}_\mu, \not{F}_\nu]\epsilon = 0 \quad (\text{B.9.13})$$

$$\not{F}_\mu = \frac{1}{2z}\Gamma_\mu{}^z - \frac{1}{2z}\partial_{\underline{i}}A\Gamma_\mu{}^i, \quad (\text{B.9.14})$$

because \not{F}_μ has no x^ν dependence.

Using the Clifford algebra commutators, we can write

$$[\Gamma_\mu{}^z, \Gamma_\nu{}^z] = -2\Gamma_{\mu\nu} \quad (\text{B.9.15})$$

$$[\Gamma_\mu{}^i, \Gamma_\nu{}^j] = -2\eta_{\mu\nu}\Gamma^{ij} - 2\delta^{ij}\Gamma_{\mu\nu} \quad (\text{B.9.16})$$

$$[\Gamma_\mu{}^z, \Gamma_\nu{}^i] = -2\eta_{\mu\nu}\Gamma^{zi} = -[\Gamma_\mu{}^i, \Gamma_\nu{}^z] \quad (\text{B.9.17})$$

with which we can simplify equation (B.9.13) to

$$-\frac{1}{2z^2}(1 + \partial_i A \partial^i A)\Gamma^{\mu\nu}\epsilon = 0. \quad (\text{B.9.18})$$

We know that $1 + \partial_i A \partial^i A \neq 0$ because $(g_S)_{ij}$ is positive definite, and that $\Gamma^{\mu\nu}\epsilon \neq 0$ because $\Gamma^{\mu\nu}$ has a trivial kernel. There are therefore no supersymmetric solutions for $k \geq 8$.

B.9.3 $k = 7$

For $k = 7$, the transverse space supports a three-form field, $G_{\underline{ijk}}e^{\underline{i}} \wedge e^{\underline{j}} \wedge e^{\underline{k}}$. Equation (B.9.1) therefore reduces to

$$\partial_\mu\epsilon - \frac{1}{2z}\Gamma_\mu{}^z\epsilon + \frac{1}{2z}\partial_{\underline{i}}A\Gamma_\mu{}^i\epsilon - \frac{A}{96z}G_{\underline{ijk}}\Gamma_\mu{}^{ijk}C * \epsilon = 0 \quad (\text{B.9.19})$$

in the $\mu = 0, \dots, k-1$ directions. In the transverse directions, G is dual to a scalar field,

$$X = \frac{1}{6}\epsilon^{ijk}G_{\underline{ijk}} \quad (\text{B.9.20})$$

$$G_{\underline{ijk}} = \epsilon_{ijk}X. \quad (\text{B.9.21})$$

Noting that $\Gamma^{ijk}C = \epsilon^{ijk}\Gamma^6$ equation (B.9.19) simplifies to

$$\partial_\mu\epsilon - \frac{1}{2z}\Gamma_\mu{}^z\epsilon + \frac{1}{2z}\partial_{\underline{i}}A\Gamma_\mu{}^i\epsilon + \frac{A}{16z}X * \Gamma_\mu{}^6\epsilon = 0 \quad (\text{B.9.22})$$

The integrability condition, equation (B.9.13), still applies, except that now

$$\not{F}_\mu = \frac{1}{2z}\Gamma_\mu{}^z - \frac{1}{2z}\partial_{\underline{i}}A\Gamma_\mu{}^i - \frac{A}{16z}X * \Gamma_\mu{}^6. \quad (\text{B.9.23})$$

Using the Clifford algebra commutators, we can write the integrability condition as

$$0 = [\not{F}_\mu, \not{F}_\nu]\epsilon \quad (\text{B.9.24})$$

$$= -\frac{1}{2z^2}\left[\left(1 + \partial_i A \partial^i A - \frac{A^2}{64}|X|^2\right)\mathbb{1} - \frac{A}{8}\partial_{\underline{i}}A X * \Gamma^{i6}\right]\Gamma_{\mu\nu}\epsilon. \quad (\text{B.9.25})$$

Note that the commutators of the Γ matrices are not identical for the $k = 6$ and $k = 7$ cases, however they differ only by terms proportional to $\eta_{\mu\nu}$. All terms symmetric in μ and ν cancel, and so we derive the same expression for either case.

As in section B.9.2, we note that $\Gamma_{\mu\nu}$ is invertible. Thus, in this case, the integrability condition is satisfied when the operator

$$\left(1 + \partial_i A \partial^i A - \frac{A^2}{64} |X|^2\right) \mathbb{1} - \frac{A}{8} \partial_{\underline{i}} A X * \Gamma^{i6} \quad (\text{B.9.26})$$

either vanishes or annihilates ϵ . The second case amounts to finding eigenvectors of $\partial_{\underline{i}} A X * \Gamma^{i6}$. We can use a rotation in the transverse dimensions to fix $\partial_{\underline{8}} A = \partial_{\underline{9}} A = 0$, without loss of generality, so that

$$\partial_{\underline{i}} A X * \Gamma^{i6} = \partial_{\underline{7}} A X \Gamma^{67} * . \quad (\text{B.9.27})$$

If an antilinear operator, such as $\Gamma^{67}*$, has any eigenvalues, then it must have real eigenvalues. $\Gamma^{67}*$ can be shown to have no real eigenvalues, and it therefore has no eigenvectors.

We are left, then, with the former case, that the operator (B.9.26) vanishes, i.e.

$$1 + \partial_i A \partial^i A - \frac{A^2}{64} |X|^2 = 0 \quad (\text{B.9.28})$$

$$\frac{A}{8} \partial_{\underline{i}} A X = 0. \quad (\text{B.9.29})$$

If X is zero then the situation is the same as the $k \geq 8$ case, and there is no solution, so from equation (B.9.29) we can conclude that $\partial_i A = 0$ and that A is therefore constant. Then, from equation (B.9.28) we find that

$$X = \frac{8}{A} e^{\varphi i}. \quad (\text{B.9.30})$$

Applying the second Killing spinor equation, equation (B.9.2), we find that

$$P_{\underline{i}} \Gamma^i C * \epsilon + \frac{1}{24} G_{\underline{ijk}} \Gamma^{ijk} \epsilon = 0. \quad (\text{B.9.31})$$

Putting this equation in terms of X and substituting using equation (B.9.30), it simplifies to

$$\left(P_{\underline{i}} \Gamma^{6i} * + \frac{2}{A} e^{\varphi i} \mathbb{1}\right) \epsilon = 0. \quad (\text{B.9.32})$$

Using a rotation in the transverse dimensions, we can fix $P_{\underline{9}} = 0$, so that

$$(P_{\underline{7}} \Gamma^{67} + P_{\underline{8}} \Gamma^{68}) * \epsilon = \frac{2}{A} e^{\varphi i} \epsilon. \quad (\text{B.9.33})$$

Because the operator on the left hand side is antilinear, the eigenvalue associated with an eigenvector is dependent on that vector's phase. Defining $\tilde{\epsilon} = e^{\frac{\varphi}{2} i} \epsilon$, we find that,

$$(P_{\underline{7}} \Gamma^{67} + P_{\underline{8}} \Gamma^{68}) * \tilde{\epsilon} = e^{-\frac{\varphi}{2} i} (P_{\underline{7}} \Gamma^{67} + P_{\underline{8}} \Gamma^{68}) * \epsilon \quad (\text{B.9.34})$$

$$= e^{-\frac{\varphi}{2} i} \frac{2}{A} e^{\varphi i} \epsilon \quad (\text{B.9.35})$$

$$= \frac{2}{A} \tilde{\epsilon} \quad (\text{B.9.36})$$

By squaring the operator on the left-hand side and rearranging, we find

$$-P_7\bar{P}_7 - P_8\bar{P}_8 + (\bar{P}_7P_8 - P_7\bar{P}_8)\Gamma^{78}\tilde{\epsilon} = \frac{4}{A^2}\tilde{\epsilon}. \quad (\text{B.9.37})$$

The eigenvalues of Γ^{78} are $\pm i$, so all eigenvalues of the squared operator take the form

$$-P_7\bar{P}_7 - P_8\bar{P}_8 \pm i(\bar{P}_7P_8 - P_7\bar{P}_8) = -(P_7 \mp iP_8)(\bar{P}_7 \pm i\bar{P}_8) \quad (\text{B.9.38})$$

$$= -(P_7 \mp iP_8)(P_7 \mp iP_8)^*. \quad (\text{B.9.39})$$

Hence, all of these eigenvalues must be negative real numbers, however equation (B.9.37) requires that the eigenvalue associated with ϵ be positive. Therefore, there are no supersymmetric solutions when $k = 7$.

B.9.4 $k = 6$

For the $k = 6$ case, it will be most convenient to work in the lightcone coordinates of appendix ???. In these coordinates, spinors will decompose as $\epsilon = \epsilon_+ + \epsilon_-$, where $\Gamma_{\pm}\epsilon_{\pm} = 0$. It will also be useful to introduce the one-form dual to G ,

$$X_{\underline{i}} = \frac{1}{6}\epsilon_{ijk\ell}G^{jk\ell} \quad (\text{B.9.40})$$

$$G_{\underline{ijk}} = -\epsilon_{ijk\ell}X^{\ell}, \quad (\text{B.9.41})$$

and to define $\Lambda_{\underline{i}}^{\pm} = \partial_{\underline{i}}A \pm \frac{A}{8}X_{\underline{i}}^*$.

In these terms, the first Killing spinor equation, (B.9.1), in the AdS directions, reduces to

$$\partial_+\epsilon + \frac{1}{2A}\left(\Gamma_+{}^z - \Lambda_{\underline{i}}^-\Gamma_+{}^i + \frac{r}{A}\partial_{\underline{i}}A\Gamma^{zi}\right)\epsilon = 0 \quad (\text{B.9.42})$$

$$\partial_-\epsilon + \frac{1}{2A}\left(\Gamma_-{}^z - \Lambda_{\underline{i}}^-\Gamma_-{}^i\right)\epsilon = 0 \quad (\text{B.9.43})$$

$$\partial_{\underline{z}}\epsilon + \frac{1}{2A}\left(\Gamma_{+-} + \Lambda_{\underline{i}}^+\Gamma^{zi} - \frac{r}{A}\partial_{\underline{i}}A\Gamma_-{}^i\right)\epsilon = 0 \quad (\text{B.9.44})$$

$$\partial_{\underline{a}}\epsilon + \frac{1}{2A}\left(\Gamma_a{}^z + \Lambda_{\underline{i}}^+\Gamma_a{}^i\right)\epsilon = 0. \quad (\text{B.9.45})$$

A simple linear transformation expresses these in terms of spacetime derivatives,

$$\partial_u\epsilon + \frac{1}{2A}\left(\Gamma_+{}^z - \Lambda_{\underline{i}}^-\Gamma_+{}^i + \frac{r}{A}\partial_{\underline{i}}A\Gamma^{zi}\right)\epsilon = 0 \quad (\text{B.9.46})$$

$$\partial_r\epsilon + \frac{1}{2A}\left(\Gamma_-{}^z - \Lambda_{\underline{i}}^-\Gamma_-{}^i\right)\epsilon = 0 \quad (\text{B.9.47})$$

$$\partial_z\epsilon + \frac{1}{2}\left(\Gamma_{+-} + \Lambda_{\underline{i}}^+\Gamma^{zi} - \frac{r}{A}\partial_{\underline{i}}A\Gamma_-{}^i\right)\epsilon - \frac{r}{A}\left(\Gamma_-{}^z - \Lambda_{\underline{i}}^-\Gamma_-{}^i\right)\epsilon = 0 \quad (\text{B.9.48})$$

$$\partial_a\epsilon + \frac{e^z}{2}\left(\Gamma_a{}^z + \Lambda_{\underline{i}}^+\Gamma_a{}^i\right)\epsilon = 0. \quad (\text{B.9.49})$$

Integrability Condition

There is only one integrability condition for the $k = 6$ case,

$$\left(1 + \Lambda_{\underline{i}}^+ \Lambda_{\underline{j}}^- \Gamma^i \Gamma^j\right) \epsilon = 0 \quad (\text{B.9.50})$$

It will be useful to introduce the operator $\Theta = \Gamma^z + \Lambda_{\underline{i}}^+ \Gamma^i$, defined such that the above integrability condition is

$$\Theta^2 \epsilon = 0. \quad (\text{B.9.51})$$

Integration in u , r , and x

Expressing equations (B.9.46) through (B.9.49) as $\partial_\mu \epsilon - \not{F}_\mu \epsilon = 0$, we see that

$$\not{F}_u = -\frac{1}{2A} \Gamma_+ \Theta \quad (\text{B.9.52})$$

$$\not{F}_r = -\frac{1}{2A} \Gamma_- \Theta \quad (\text{B.9.53})$$

$$\not{F}_a = -\frac{e^z}{2} \Gamma_a \Theta. \quad (\text{B.9.54})$$

Thus, the solution in the u -, r -, and x -directions is

$$\epsilon(u, r, z, x^a, y^i) = e^{u \not{F}_u + r \not{F}_r + x^a \not{F}_a} \epsilon(0, 0, z, 0, y^i) \quad (\text{B.9.55})$$

$$= \left[1 - \left(\frac{u}{2A} \Gamma_+ + \frac{r}{2A} \Gamma_- + \frac{e^z}{2} x^a \Gamma_a \right) \Theta \right] \epsilon(0, 0, z, 0, y^i). \quad (\text{B.9.56})$$

Integration in z

Expressing equation (B.9.48) as $\partial_z \epsilon - \not{F}_z \epsilon = 0$, \not{F}_z is

$$\not{F}_z = -\frac{1}{2} \left(\Gamma_{+-} + \Lambda_{\underline{i}}^+ \Gamma^{zi} \right) - \frac{r}{A} \Gamma_- \Theta. \quad (\text{B.9.57})$$

Restricting this equation to the $r = 0$ hyperplane, we can reduce \not{F}_z to

$$\not{F}_z = -\frac{1}{2} \left(\Gamma_{+-} + \Lambda_{\underline{i}}^+ \Gamma^{zi} \right). \quad (\text{B.9.58})$$

Γ_{+-} has the property that $\Gamma_{+-} \epsilon_\pm = \pm \epsilon_\pm$, so the action of \not{F}_z on ϵ_\pm is

$$\not{F}_z \epsilon_+ = -\frac{1}{2} \Gamma^z \Theta \epsilon_+ \quad (\text{B.9.59})$$

$$\not{F}_z \epsilon_- = -\frac{1}{2} (\Gamma^z \Theta - 2) \epsilon_-. \quad (\text{B.9.60})$$

When we square $\Gamma^z \Theta$, we get

$$\Gamma^z \Theta \Gamma^z \Theta \epsilon = \left(1 + \Lambda_{\underline{i}}^+ \Gamma^{zi} \right) \left(1 + \Lambda_{\underline{j}}^+ \Gamma^{zj} \right) \epsilon \quad (\text{B.9.61})$$

$$= \left(1 + 2\Lambda_{\underline{i}}^+ \Gamma^{zi} - \Lambda_{\underline{i}}^+ \Lambda_{\underline{j}}^- \Gamma^i \Gamma^j \right) \epsilon \quad (\text{B.9.62})$$

$$= 2 \left(1 + \Lambda_{\underline{i}}^+ \Gamma^{zi} \right) \epsilon \quad (\text{B.9.63})$$

$$= 2\Gamma^z \Theta \quad (\text{B.9.64})$$

We can therefore simplify the solution in the z -direction,

$$\epsilon(0, 0, z, 0, y^i) = e^{z\mathcal{F}^z} \eta(y^i), \quad (\text{B.9.65})$$

by evaluating $\mathcal{F}_z^n \epsilon_\pm$,

$$\mathcal{F}_z^n \epsilon_+ = \left(-\frac{1}{2} \Gamma^z \Theta \right)^n \epsilon_+ \quad (\text{B.9.66})$$

$$= \frac{(-1)^n}{2} \Gamma^z \Theta \epsilon_+ \quad (\text{B.9.67})$$

$$\mathcal{F}_z^n \epsilon_- = \left[-\frac{1}{2} (\Gamma^z \Theta - 2) \right]^n \epsilon_- \quad (\text{B.9.68})$$

$$= -\frac{1}{2} (\Gamma^z \Theta - 2) \epsilon_- \quad (\text{B.9.69})$$

ϵ_+ is therefore

$$\epsilon_+(0, 0, z, 0, y^i) = \left(1 + \frac{e^{-z} - 1}{2} \Gamma^z \Theta \right) \eta_+(y^i) \quad (\text{B.9.70})$$

while ϵ_- is

$$\epsilon_-(0, 0, z, 0, y^i) = \left(e^z - \frac{e^z - 1}{2} \Gamma^z \Theta \right) \eta_-(y^i). \quad (\text{B.9.71})$$

Together, these can be expressed as

$$\epsilon(0, 0, z, 0, y^i) = \left(1 + \frac{e^{-z} - 1}{2} \Gamma^z \Theta \right) e^{\mathcal{P}^- z} \eta(y^i). \quad (\text{B.9.72})$$

Composition

To find the full solution for ϵ , we will need to combine the results of the two previous sections. It will help to note that

$$\Theta \epsilon_+(0, 0, z, 0, y^i) = \left(\Theta + \frac{e^{-z} - 1}{2} \Theta \Gamma^z \Theta \right) \eta_+ \quad (\text{B.9.73})$$

$$= e^{-z} \Theta \eta_+ \quad (\text{B.9.74})$$

and

$$\Theta \epsilon_-(0, 0, z, 0, y^i) = \left(e^z \Theta - \frac{e^z - 1}{2} \Theta \Gamma^z \Theta \right) \eta_- \quad (\text{B.9.75})$$

$$= \Theta \eta_-. \quad (\text{B.9.76})$$

Thus, the full solutions for ϵ_+ and ϵ_- are

$$\epsilon_+(u, r, z, x^a, y^i) = \left(1 - \frac{e^z}{2} x^a \Gamma_a \Theta \right) \epsilon_+(0, 0, z, 0, y^i) - \frac{u}{2A} \Gamma_+ \Theta \epsilon_-(0, 0, z, 0, y^i) \quad (\text{B.9.77})$$

$$= \left(1 + \frac{e^{-z} - 1}{2} \Gamma^z \Theta \right) \eta_+ - \frac{x^a}{2} \Gamma_a \Theta \eta_+ - \frac{u}{2A} \Gamma_+ \Theta \eta_- \quad (\text{B.9.78})$$

$$= \left[1 + \left(\frac{e^{-z} - 1}{2} \Gamma^z - \frac{x^a}{2} \Gamma_a \right) \Theta \right] \eta_+ - \frac{u}{2A} \Gamma_+ \Theta \eta_- \quad (\text{B.9.79})$$

and

$$\epsilon_{-}(u, r, z, x^a, y^i) = \left[1 - \frac{e^z}{2} x^a \Gamma_a \Theta \right] \epsilon_{-}(0, 0, z, 0, y^i) - \frac{r}{2A} \Gamma_{-} \Theta \epsilon_{+}(0, 0, z, 0, y^i) \quad (\text{B.9.80})$$

$$= \left(e^z - \frac{e^z - 1}{2} \Gamma^z \Theta \right) \eta_{-} - \frac{e^z}{2} x^a \Gamma_a \Theta \eta_{-} - \frac{e^{-z}}{2A} r \Gamma_{-} \Theta \eta_{+} \quad (\text{B.9.81})$$

$$= e^z \left[1 + \left(\frac{e^{-z} - 1}{2} \Gamma^z - \frac{x^a}{2} \Gamma_a \right) \Theta \right] \eta_{-} - \frac{e^{-z}}{2A} r \Gamma_{-} \Theta \eta_{+}. \quad (\text{B.9.82})$$

Algebraic Condition

Multiplying equation (B.9.2) by $C*$, we can write the algebraic Killing spinor equation in terms of X ,

$$\begin{aligned} 0 &= \bar{P}_{\underline{i}} \Gamma^i \epsilon + \frac{1}{24} \bar{G}_{\underline{ijk}} \Gamma^{ijk} C * \epsilon \\ &= \bar{P}_{\underline{i}} \Gamma^i \epsilon + \frac{1}{4} \bar{X}_{\underline{i}} \Gamma^i * \epsilon. \end{aligned} \quad (\text{B.9.83})$$

applying this to the result from section B.9.4, we find two algebraic conditions on $\eta(y^i)$,

$$0 = \bar{P}_{\underline{i}} \Gamma^i \eta + \frac{1}{4} \bar{X}_{\underline{i}} \Gamma^i * \eta \quad (\text{B.9.84})$$

$$0 = \bar{P}_{\underline{i}} \Gamma^i \Theta \eta + \frac{1}{4} \bar{X}_{\underline{i}} \Gamma^i * \Theta \eta. \quad (\text{B.9.85})$$

Transverse Dimensions

In the transverse dimensions, the covariant derivative includes two AdS-direction components,

$$\nabla_{\underline{i}} \epsilon = \left(e_{\underline{i}}^j \partial_j + \Omega_{\underline{i}, \underline{jk}} \Gamma^{jk} + \Omega_{\underline{i}, \underline{\mu\nu}} \Gamma^{\mu\nu} \right) \epsilon \quad (\text{B.9.86})$$

$$= \left(e_{\mathcal{S}\underline{i}}^j \partial_j + \Omega_{\mathcal{S}\underline{i}, \underline{jk}} \Gamma^{jk} + \frac{2r}{A} \partial_{\underline{i}} A \partial_r + \frac{1}{2A} \partial_{\underline{i}} A \Gamma_{+-} \right) \epsilon \quad (\text{B.9.87})$$

$$= \tilde{\nabla}_{\underline{i}} \epsilon - e^{-z} \frac{r}{A^2} \partial_{\underline{i}} A \Gamma_{-} \Theta \epsilon + \frac{1}{2A} \partial_{\underline{i}} A \Gamma_{+-} \epsilon, \quad (\text{B.9.88})$$

where $\tilde{\nabla}_{\underline{i}}$ is the covariant derivative on \mathcal{S} considered as a submanifold of the full space. The first Killing spinor equation, (B.9.1), is therefore

$$\left(\tilde{\nabla}_{\underline{i}} - e^{-z} \frac{r}{A^2} \Gamma_{-} \Theta + \frac{1}{2A} \partial_{\underline{i}} A \Gamma_{+-} - \frac{i}{2} Q_{\underline{i}} \right) \epsilon + \left(-\frac{1}{16} X_{\underline{i}} * + \frac{3}{16} X_{\underline{j}} \Gamma_i^j * \right) \epsilon = 0. \quad (\text{B.9.89})$$

Computations in the transverse directions will be done at the AdS origin, $x^\mu = 0$, so we will define $\not{F}_{\underline{i}}$ as

$$\not{F}_{\underline{i}} = -\frac{1}{2A} \partial_{\underline{i}} A \Gamma_{+-} + \frac{i}{2} Q_{\underline{i}} - \frac{1}{16} X_{\underline{i}} * + \frac{3}{16} X_{\underline{j}} \Gamma_i^j *. \quad (\text{B.9.90})$$

so equation (B.9.89) can be written as

$$\tilde{\nabla}_{\underline{i}} \epsilon - \not{F}_{\underline{i}} \epsilon - e^{-z} \frac{r}{A^2} \Gamma_{-} \Theta \epsilon = 0. \quad (\text{B.9.91})$$

Applying this to equations (B.9.79) and (B.9.82), we find the differential Killing spinors restricted to the transverse space,

$$\tilde{\nabla}_{\underline{i}}\eta - \not{F}_{\underline{i}}\eta = 0, \quad (\text{B.9.92})$$

as well as several additional integrability conditions. By setting $u = r = z = 0$, the first integrability condition is found to be

$$\tilde{\nabla}_{\underline{i}}(\Theta\eta) - \not{F}_{\underline{i}}(\Theta\eta) = 0, \quad (\text{B.9.93})$$

or, equivalently,

$$\begin{aligned} 0 = & \left(\tilde{\nabla}_{\underline{i}}\tilde{\nabla}_{\underline{j}}A\Gamma^j - \frac{1}{8}\tilde{\nabla}_{\underline{i}}AX_{\underline{j}}\Gamma^j * - \frac{A}{8}\tilde{\nabla}_{\underline{i}}X_{\underline{j}}\Gamma^j * \right)\eta \\ & + \left(\frac{iA}{8}Q_{\underline{i}}X_{\underline{k}}\Gamma^k * - \frac{1}{8}\tilde{\nabla}_{\underline{k}}AX_{\underline{i}}\Gamma^k * + \frac{A}{64}X_{(\underline{i}}\bar{X}_{\underline{k})}\Gamma^k \right. \\ & \left. - \frac{3}{8}\tilde{\nabla}_{\underline{k}}AX_{\underline{j}}\Gamma^{jk} - \frac{3A}{64}X_{[\underline{i}}\bar{X}_{\underline{j}]}\Gamma^j \right)\eta. \end{aligned} \quad (\text{B.9.94})$$

Similarly, we can set $r = z = x^a = 0$ or $u = z = x^a = 0$ to find that

$$\tilde{\nabla}_{\underline{i}}\left(\frac{1}{A}\Gamma_+\Theta\eta_-\right) - \not{F}_{\underline{i}}\left(\frac{1}{A}\Gamma_+\Theta\eta_-\right) = 0 \quad (\text{B.9.95})$$

and

$$\tilde{\nabla}_{\underline{i}}\left(\frac{1}{A}\Gamma_-\Theta\eta_+\right) - \not{F}_{\underline{i}}\left(\frac{1}{A}\Gamma_-\Theta\eta_+\right) + \frac{2}{A^2}\Gamma_-\Theta\eta_+ = 0, \quad (\text{B.9.96})$$

respectively. The Γ matrices commute with the covariant derivative, so the latter conditions reduce to

$$\frac{1}{A^2}\partial_{\underline{i}}A\Gamma_+\Theta\eta_- + \left[\not{F}_{\underline{i}}, \frac{1}{A}\Gamma_+\right]\Theta\eta_- = 0 \quad (\text{B.9.97})$$

$$\frac{1}{A^2}\partial_{\underline{i}}A\Gamma_-\Theta\eta_+ + \left[\not{F}_{\underline{i}}, \frac{1}{A}\Gamma_-\right]\Theta\eta_+ = \frac{2}{A^2}\Gamma_-\Theta\eta_+. \quad (\text{B.9.98})$$

From the form of $\not{F}_{\underline{i}}$, these conditions are automatically satisfied. The only independent mixed integrability condition is therefore equation (B.9.94).

We can derive the purely-transverse integrability conditions by expanding the commutator $[\tilde{\nabla}_{\underline{i}}, \tilde{\nabla}_{\underline{j}}]\eta$ with equation (B.9.92),

$$[\tilde{\nabla}_{\underline{i}}, \tilde{\nabla}_{\underline{j}}]\eta = \tilde{\nabla}_{\underline{i}}\tilde{\nabla}_{\underline{j}}\eta - \tilde{\nabla}_{\underline{j}}\tilde{\nabla}_{\underline{i}}\eta \quad (\text{B.9.99})$$

$$= \tilde{\nabla}_{\underline{i}}(\not{F}_{\underline{j}}\eta) - \tilde{\nabla}_{\underline{j}}(\not{F}_{\underline{i}}\eta) \quad (\text{B.9.100})$$

$$\frac{1}{4}\tilde{R}_{S\underline{ij},\underline{k}\underline{\ell}}\Gamma^{k\underline{\ell}}\eta = 2\tilde{\nabla}_{[\underline{i}}\not{F}_{\underline{j}]}\eta - [\not{F}_{\underline{i}}, \not{F}_{\underline{j}}]\eta. \quad (\text{B.9.101})$$

The right hand side expands to

$$\begin{aligned}
2\tilde{\nabla}_{[\underline{i}}\not{F}_{\underline{j}]} - [\not{F}_{\underline{i}}, \not{F}_{\underline{j}}] &= \frac{i}{2}(dQ)_{\underline{i}\underline{j}} - \frac{1}{16}(dX)_{\underline{i}\underline{j}}* + \frac{3}{16}\left(\tilde{\nabla}_{\underline{i}}X_{\underline{k}}\Gamma_j{}^k - \tilde{\nabla}_{\underline{j}}X_{\underline{k}}\Gamma_i{}^k\right)* \\
&+ \frac{i}{8}Q_{[\underline{i}}X_{\underline{j}]}* - \frac{3i}{8}X_{\underline{k}}Q_{[\underline{i}}\Gamma_{\underline{j}]}{}^k* + \frac{1}{16}X_{[\underline{i}}\overline{X}_{\underline{j}]} + \frac{9}{128}|X|^2\Gamma_{ij} \\
&+ \frac{3}{128}X_{[\underline{i}}\overline{X}_{\underline{k}]} \Gamma_j{}^k - \frac{3}{128}X_{[\underline{j}}\overline{X}_{\underline{k}]} \Gamma_i{}^k \\
&+ \frac{9}{128}X_{(\underline{i}}\overline{X}_{\underline{k})} \Gamma_j{}^k - \frac{9}{128}X_{(\underline{j}}\overline{X}_{\underline{k})} \Gamma_i{}^k \\
&+ \frac{9}{128}X_{\underline{k}}\overline{X}_{\underline{\ell}} \Gamma_{ij}{}^{k\ell}.
\end{aligned} \tag{B.9.102}$$

Bosonic Field Equations and Bianchi Identities

The Einstein equation,

$$R_{\underline{M}\underline{N}} - \frac{1}{4}G_{(\underline{M}}{}^{\underline{K}_1\underline{K}_2}\overline{G}_{\underline{N})\underline{K}_1\underline{K}_2} + \frac{1}{48}G_{\underline{K}_1\underline{K}_2\underline{K}_3}\overline{G}^{\underline{K}_1\underline{K}_2\underline{K}_3}\eta_{\underline{M}\underline{N}} - 2P_{(\underline{M}}\overline{P}_{\underline{N})} = 0 \tag{B.9.103}$$

simplifies, in the AdS-directions, to

$$\left[-\frac{5}{2A^2}(1+|dA|^2) + \frac{1}{A^2}|dA|^2 - \frac{1}{A}\tilde{\nabla}^2 A\right]\eta_{\mu\nu} + \frac{1}{8}|X|^2\eta_{\mu\nu} = 0 \tag{B.9.104}$$

which tells us that

$$|X|^2 = -\frac{20}{A^2}(1+|dA|^2) + \frac{8}{A^2}|dA|^2 - \frac{8}{A}\tilde{\nabla}^2 A. \tag{B.9.105}$$

In the \mathcal{S} -directions, using $G_{(\underline{i}}{}^{\underline{k}_1\underline{k}_2}\overline{G}_{\underline{j})\underline{k}_1\underline{k}_2} = 2|X|^2\delta_{ij} - 2X_{(\underline{i}}\overline{X}_{\underline{j})}$, the Einstein equation instead reduces to

$$\tilde{R}_{\mathcal{S}\underline{i}\underline{j}} + \frac{6}{A^2}\partial_{\underline{i}}A\partial_{\underline{j}}A - \frac{6}{A}\tilde{\nabla}_{\underline{i}}\tilde{\nabla}_{\underline{j}}A - \frac{3}{8}|X|^2\delta_{ij} + \frac{1}{2}X_{(\underline{i}}\overline{X}_{\underline{j})} + 2P_{(\underline{i}}\overline{P}_{\underline{j})} = 0. \tag{B.9.106}$$

Expressed in terms of X , the remaining field equations and Bianchi identities are

$$0 = \tilde{\nabla}_i X^i - iQ_i X^i + P_i \overline{X}^i \tag{B.9.107}$$

$$0 = dP - 2iQ \wedge P \tag{B.9.108}$$

$$0 = dQ + iP \wedge \overline{P} \tag{B.9.109}$$

$$0 = dX + \frac{6}{A}dA \wedge X - iQ \wedge X - P \wedge \overline{X} \tag{B.9.110}$$

$$0 = \tilde{\nabla}^i P_i + \frac{6}{A}\partial^i A P_i - 2iQ^i P_i + \frac{1}{4}|X|^2 \tag{B.9.111}$$

Reduction to \mathcal{S}

We can apply these equations to the purely transverse integrability condition in section B.9.4 by multiplying the condition by Γ^j . The derivative is thus

$$\begin{aligned} 2\Gamma^j \tilde{\nabla}_{[\underline{i}} \not{F}_{\underline{j}]} &= \frac{i}{2} (dQ)_{\underline{i}\underline{j}} \Gamma^j - \frac{1}{16} (dX)_{\underline{i}\underline{j}} \Gamma^j * + \frac{3}{8} \tilde{\nabla}_{\underline{i}} X_{\underline{j}} \Gamma^j * \\ &\quad + \frac{3}{8} (dX)_{\underline{j}\underline{k}} \Gamma^{jk} * + \frac{3}{16} \tilde{\nabla}_{\underline{j}} X^j \Gamma_i *, \end{aligned} \quad (\text{B.9.112})$$

$$\begin{aligned} &= P_{[\underline{i}} \bar{P}_{\underline{j}]} \Gamma^j + \frac{3}{4A} \tilde{\nabla}_{[\underline{i}} A X_{\underline{j}]} \Gamma^j * - \frac{i}{8} Q_{[\underline{i}} X_{\underline{j}]} \Gamma^j * - \frac{1}{8} P_{[\underline{i}} \bar{X}_{\underline{j}]} \Gamma^j * \\ &\quad - \frac{9}{2A} \tilde{\nabla}_{\underline{j}} A X_{\underline{k}} \Gamma^{jk} * + \frac{3i}{4} Q_{\underline{j}} X_{\underline{k}} \Gamma^{jk} * + \frac{3}{4} P_{\underline{j}} X_{\underline{k}} \Gamma^{jk} * \\ &\quad + \frac{3i}{16} Q_j X^j \Gamma_i * - \frac{3}{16} P_j \bar{X}^j \Gamma_i * + \frac{3}{8} \tilde{\nabla}_{\underline{i}} X_{\underline{j}} \Gamma^j *, \end{aligned} \quad (\text{B.9.113})$$

while the commutator is

$$\begin{aligned} -\Gamma^j [\not{F}_{\underline{i}}, \not{F}_{\underline{j}}] &= \frac{i}{8} Q_{[\underline{i}} X_{\underline{j}]} \Gamma^j * - \frac{3i}{8} Q_{\underline{i}} X_{\underline{k}} \Gamma^k * - \frac{3i}{16} X_{\underline{k}} Q_{\underline{j}} \Gamma^{jk} * - \frac{3i}{16} X_j Q^j \Gamma_i * \\ &\quad + \frac{7}{64} X_{[\underline{i}} \bar{X}_{\underline{j}]} \Gamma^j - \frac{9}{64} |X|^2 \Gamma_i - \frac{3}{64} X_{\underline{j}} \bar{X}_{\underline{k}} \Gamma^{jk} \\ &\quad + \frac{9}{64} X_{(\underline{i}} \bar{X}_{\underline{k})} \Gamma^k, \end{aligned} \quad (\text{B.9.114})$$

and the curvature term becomes

$$\frac{1}{2} \tilde{R}_{Sij} \Gamma^j = -\frac{3}{A^2} \tilde{\nabla}_{\underline{i}} A \tilde{\nabla}_{\underline{j}} A \Gamma^j + \frac{3}{A} \tilde{\nabla}_{\underline{i}} \tilde{\nabla}_{\underline{j}} A \Gamma^j + \frac{3}{16} |X|^2 \Gamma_i - \frac{1}{4} X_{(\underline{i}} \bar{X}_{\underline{j})} \Gamma^j - P_{(\underline{i}} \bar{P}_{\underline{j})} \Gamma^j.$$

The integrability condition is therefore

$$\begin{aligned} 0 &= \left(-P_{\underline{j}} \bar{P}_{\underline{i}} \Gamma^j + \frac{3}{4A} \tilde{\nabla}_{[\underline{i}} A X_{\underline{j}]} \Gamma^j * - \frac{9}{2A} \tilde{\nabla}_{\underline{j}} A X_{\underline{k}} \Gamma^{jk} * - \frac{1}{8} P_{[\underline{i}} \bar{X}_{\underline{j}]} \Gamma^j * \right. \\ &\quad - \frac{3i}{8} Q_{\underline{i}} X_{\underline{k}} \Gamma^k * + \frac{15i}{16} Q_{\underline{j}} X_{\underline{k}} \Gamma^{jk} * - \frac{3}{16} P_j \bar{X}^j \Gamma_i * + \frac{3}{4} P_{\underline{j}} X_{\underline{k}} \Gamma^{jk} * \\ &\quad + \frac{3}{8} \tilde{\nabla}_{\underline{i}} X_{\underline{j}} \Gamma^j * - \frac{7}{64} X_{\underline{j}} \bar{X}_{\underline{i}} \Gamma^j + \frac{3}{64} |X|^2 \Gamma_i - \frac{3}{64} X_{\underline{j}} \bar{X}_{\underline{k}} \Gamma^{jk} \\ &\quad \left. - \frac{3}{A^2} \tilde{\nabla}_{\underline{i}} A \tilde{\nabla}_{\underline{j}} A \Gamma^j + \frac{3}{A} \tilde{\nabla}_{\underline{i}} \tilde{\nabla}_{\underline{j}} A \Gamma^j \right) \eta \end{aligned} \quad (\text{B.9.115})$$

Lichnerowicz Theorem(s)

We wish to prove that, on the transverse manifold, \mathcal{S} , the Dirac equation is equivalent to the Killing spinor equations. As such, we will assume that the Dirac equations,

$$0 = \mathcal{D}^{(\pm)} \eta_{\pm} \quad (\text{B.9.116})$$

$$= \Gamma^i \tilde{\nabla}_{\underline{i}} \eta_{\pm} - \Gamma^i \not{F}_{\underline{i}}^{(\pm)} \eta \quad (\text{B.9.117})$$

hold, where

$$\not{F}_{\underline{i}}^{(\pm)} = \mp \frac{1}{2A} \partial_{\underline{i}} A + \frac{i}{2} Q_{\underline{i}} - \frac{1}{16} X_{\underline{i}} * + \frac{3}{16} X_{\underline{j}} \Gamma_i^j *, \quad (\text{B.9.118})$$

and prove that the Killing spinor equations must hold as well.

To this end, we begin by computing the following Laplacian in the transverse dimensions,

$$\tilde{\nabla}^2 \langle \eta_{\pm}, \eta_{\pm} \rangle = 2 \operatorname{Re} \langle \eta_{\pm}, \tilde{\nabla}^2 \eta_{\pm} \rangle + 2 \langle \tilde{\nabla}_i \eta_{\pm}, \tilde{\nabla}^i \eta_{\pm} \rangle. \quad (\text{B.9.119})$$

We expand the first term using

$$\tilde{\nabla}^i \tilde{\nabla}_i \eta_{\pm} = \Gamma^i \tilde{\nabla}_{\underline{i}} \left(\Gamma^j \tilde{\nabla}_{\underline{j}} \eta_{\pm} \right) + \frac{1}{4} \tilde{R}_S \eta_{\pm} \quad (\text{B.9.120})$$

$$\begin{aligned} &= \Gamma^i \tilde{\nabla}_{\underline{i}} \left(\Gamma^j \not{F}_{\underline{j}}^{(\pm)} \right) \eta_{\pm} + \Gamma^i \Gamma^j \not{F}_{\underline{j}}^{(\pm)} \tilde{\nabla}_{\underline{i}} \eta_{\pm} - \frac{3}{2A^2} |dA|^2 \eta_{\pm} - \frac{3}{2A} \tilde{\nabla}^2 A \eta_{\pm} \\ &\quad - \frac{1}{4} |X|^2 \eta_{\pm} + \frac{1}{2} |P|^2 \eta_{\pm}, \end{aligned} \quad (\text{B.9.121})$$

which, along with

$$\Gamma^i \not{F}_{\underline{i}}^{(\pm)} = \mp \frac{1}{2A} \partial_{\underline{i}} A \Gamma^i + \frac{i}{2} Q_{\underline{i}} \Gamma^i + \frac{1}{2} X_{\underline{i}} \Gamma^i *, \quad (\text{B.9.122})$$

implies that

$$\begin{aligned} \operatorname{Re} \langle \eta_{\pm}, \tilde{\nabla}^2 \eta_{\pm} \rangle &= \pm \left(\frac{1}{2A^2} |dA|^2 - \frac{1}{2A} \tilde{\nabla}^2 A \right) \langle \eta_{\pm}, \eta_{\pm} \rangle + \left\langle \eta_{\pm}, \frac{i}{4} (dQ)_{\underline{i}\underline{j}} \Gamma^{ij} \eta_{\pm} \right\rangle \\ &\quad + \operatorname{Re} \left\langle \eta_{\pm}, \left(\frac{1}{4} (dX)_{\underline{i}\underline{j}} \Gamma^{ij} + \tilde{\nabla}^i X_i \right) * \eta_{\pm} \right\rangle \\ &\quad + \operatorname{Re} \left\langle \eta_{\pm}, \Gamma^i \left(\mp \frac{1}{2A} \partial_{\underline{i}} A \Gamma^j + \frac{i}{2} Q_{\underline{j}} \Gamma^j + \frac{1}{2} X_{\underline{j}} \Gamma^j * \right) \tilde{\nabla}_{\underline{i}} \eta_{\pm} \right\rangle \\ &\quad - \left(\frac{3}{2A^2} |dA|^2 - \frac{3}{2A} \tilde{\nabla}^2 A - \frac{1}{4} |X|^2 + \frac{1}{2} |P|^2 \right) \langle \eta_{\pm}, \eta_{\pm} \rangle \end{aligned} \quad (\text{B.9.123})$$

$$\begin{aligned} &= \left\langle \eta_{\pm}, \frac{1}{2} P_{[\underline{i}} \overline{P}_{\underline{j}]} \Gamma^{ij} \eta_{\pm} \right\rangle + \operatorname{Re} \langle \eta_{\pm}, (iQ_i X^i - P_i X^i) * \eta_{\pm} \rangle \\ &\quad + \operatorname{Re} \left\langle \eta_{\pm}, \Gamma^i \left(\mp \frac{1}{2A} \partial_{\underline{i}} A \Gamma^j + \frac{i}{2} Q_{\underline{j}} \Gamma^j + \frac{1}{2} X_{\underline{j}} \Gamma^j * \right) \tilde{\nabla}_{\underline{i}} \eta_{\pm} \right\rangle \\ &\quad - \left(\frac{3 \mp 1}{2A^2} |dA|^2 - \frac{3 \mp 1}{2A} \tilde{\nabla}^2 A - \frac{1}{4} |X|^2 + \frac{1}{2} |P|^2 \right) \langle \eta_{\pm}, \eta_{\pm} \rangle \end{aligned} \quad (\text{B.9.124})$$

noting that

$$\langle \eta_{\pm}, \Gamma^{ij} * \eta_{\pm} \rangle = 0. \quad (\text{B.9.125})$$

For the second term, we can expand it by completing the square,

$$\begin{aligned} \langle \tilde{\nabla}_i \eta_{\pm}, \tilde{\nabla}^i \eta_{\pm} \rangle &= \left\langle \tilde{\nabla}_i \eta_{\pm} - \not{F}_i^{(\pm)} \eta_{\pm}, \tilde{\nabla}^i \eta_{\pm} - \not{F}^{(\pm)i} \eta_{\pm} \right\rangle - \left\langle \not{F}_i^{(\pm)} \eta_{\pm}, \not{F}^{(\pm)i} \eta_{\pm} \right\rangle \\ &\quad - 2 \operatorname{Re} \left\langle \eta_{\pm}, \left(\pm \frac{1}{2A} \partial_{\underline{i}} A + \frac{i}{2} Q_{\underline{i}} + \frac{1}{16} X_{\underline{i}} * + \frac{3}{16} X_{\underline{j}} \Gamma_i^j * \right) \tilde{\nabla}^i \eta_{\pm} \right\rangle. \end{aligned} \quad (\text{B.9.126})$$

B.10 IIB AdS Directly From Horizon

B.10.1 Maximality Condition on η_+

We know that, if $\mathcal{D}_8^{(+)}\eta_{8+} = 0$, then

$$\tilde{\nabla}_8^i \tilde{\nabla}_{8i} \|\eta_{8+}\|^2 - h^i \tilde{\nabla}_{8i} \|\eta_{8+}\|^2 = 2 \left\| \nabla_8^{(+)} \eta_{8+} \right\|^2 + \left\| \mathcal{A}^{(+)} \eta_{8+} \right\|^2, \quad (\text{B.10.1})$$

where

$$\mathcal{D}_8^{(+)} = \Gamma^z \nabla_{8z}^{(+)} + \Gamma^a \nabla_{8a}^{(+)} + \mathcal{D}^{(+)} \quad (\text{B.10.2})$$

$$\tilde{\nabla}_8^i \tilde{\nabla}_{8i} \phi = \frac{1}{A^2} \partial_z \partial_z \phi + \frac{e^{-2z}}{A^2} \delta^{ab} \partial_a \partial_b \phi + \frac{4}{A} \partial^i A \partial_i \phi + \frac{3}{A^2} \partial_z \phi + \tilde{\nabla}^2 \phi \quad (\text{B.10.3})$$

$$h = -2dz - \frac{2}{A} dA \quad (\text{B.10.4})$$

$$h^i \tilde{\nabla}_{8i} \phi = -\frac{2}{A} \partial_z \phi - \frac{2}{A} \partial^i A \partial_i \phi \quad (\text{B.10.5})$$

$$\nabla_{8z}^{(+)} \psi = \partial_z \psi + \frac{1}{2A} \Gamma_z \Theta \psi \quad (\text{B.10.6})$$

$$\nabla_{8a}^{(+)} \psi = \partial_a \psi + \frac{1}{2A} \Gamma_a \Theta \psi \quad (\text{B.10.7})$$

$$\mathcal{A}^{(+)} = \bar{P}_i \Gamma^i + \frac{1}{4} \bar{X}_i \Gamma^i \star \quad (\text{B.10.8})$$

and

$$\begin{aligned} \eta_{8+} &= \epsilon_+ (0, 0, z, x^a, y^i), \\ &= \left[1 + \left(\frac{e^{-z} - 1}{2} \Gamma_z - \frac{x^a}{2} \Gamma_a \right) \Theta \right] \eta_+ (y^i). \end{aligned} \quad (\text{B.10.9})$$

If we assume that ϵ_+ satisfies the pure-AdS integrability condition, $\Theta^2 \epsilon_+ = 0$, as well as a weaker form of the mixed AdS-transverse integrability condition, $\mathcal{D}^{(+)} \epsilon_+ = 0$, then $\mathcal{D}_8^{(+)} \eta_{8+} = \mathcal{D}^{(+)} \eta$ is the Dirac operator on the transverse space,

$$\left\| \nabla_8^{(+)} \eta_{8+} \right\|^2 = \left\| \nabla^{(+)} \eta_+ + \left(\frac{e^{-z} - 1}{2} \Gamma_z - \frac{x^a}{2} \Gamma_a \right) \nabla^{(+)} (\Theta \eta_+) \right\|^2 \quad (\text{B.10.10})$$

$$\begin{aligned} &= \left\| \nabla^{(+)} \eta_+ \right\|^2 + \frac{1}{4} \left[(e^{-z} - 1)^2 + |x|^2 \right] \left\| \nabla^{(+)} (\Theta \eta_+) \right\|^2 \\ &\quad + 2 \operatorname{Re} \left\langle \nabla_i^{(+)} \eta_+, \left(\frac{e^{-z} - 1}{2} \Gamma_z - \frac{x^a}{2} \Gamma_a \right) \nabla^{(+i)} (\Theta \eta_+) \right\rangle, \end{aligned} \quad (\text{B.10.11})$$

and

$$\left\| \mathcal{A}^{(+)} \eta_{8+} \right\|^2 = \left\| \mathcal{A}^{(+)} \eta_+ - \left(\frac{e^{-z} - 1}{2} \Gamma_z - \frac{x^a}{2} \Gamma_a \right) \mathcal{A}^{(+)} \Theta \eta_+ \right\|^2 \quad (\text{B.10.12})$$

$$\begin{aligned} &= \left\| \mathcal{A}^{(+)} \eta_+ \right\|^2 + \frac{1}{4} \left[(e^{-z} - 1)^2 + |x|^2 \right] \left\| \mathcal{A}^{(+)} \Theta \eta_+ \right\|^2 \\ &\quad - 2 \operatorname{Re} \left\langle \mathcal{A}^{(+)} \eta_+, \left(\frac{e^{-z} - 1}{2} \Gamma_z - \frac{x^a}{2} \Gamma_a \right) \mathcal{A}^{(+)} \Theta \eta_+ \right\rangle, \end{aligned} \quad (\text{B.10.13})$$

On the left hand side,

$$\begin{aligned}\|\eta_{8+}\|^2 &= \|\eta_+\|^2 + \frac{1}{4}[(e^{-z} - 1)^2 + |x|^2]\|\Theta\eta_+\|^2 \\ &\quad + 2\operatorname{Re}\left\langle\eta_+, \left(\frac{e^{-z}-1}{2}\Gamma_z + \frac{x^a}{2}\Gamma_a\right)\Theta\eta_+\right\rangle,\end{aligned}\tag{B.10.14}$$

so

$$\begin{aligned}\tilde{\nabla}_8^i \tilde{\nabla}_{8i} \|\eta_{8+}\|^2 - h^i \tilde{\nabla}_{8i} \|\eta_{8+}\|^2 &= \frac{1}{A^2} \partial_z \partial_z \|\eta_{8+}\|^2 + \frac{e^{-2z}}{A^2} \delta^{ab} \partial_a \partial_b \|\eta_{8+}\|^2 \\ &\quad + \frac{6}{A} \partial^i A \partial_i \|\eta_{8+}\|^2 + \frac{5}{A^2} \partial_z \|\eta_{8+}\|^2 + \tilde{\nabla}^2 \|\eta_{8+}\|^2\end{aligned}\tag{B.10.15}$$

$$\begin{aligned}&= -\frac{1}{2A^2} e^{-z} (3e^{-z} - 4) \|\Theta\eta_+\|^2 - \frac{4e^{-z}}{A^2} \operatorname{Re}\langle\eta_+, \Gamma_z \Theta\eta_+\rangle \\ &\quad + \frac{3e^{-2z}}{2A^2} \|\Theta\eta_+\|^2 + \frac{6}{A} \partial^i A \partial_i \|\eta_+\|^2 \\ &\quad + \frac{3}{2A} [(e^{-z} - 1)^2 + |x|^2] \partial^i A \partial_i \|\Theta\eta_+\|^2 \\ &\quad + \frac{6}{A} \partial^i A \partial_i \operatorname{Re}\langle\eta_+, [(e^{-z} - 1)\Gamma_z + x^a \Gamma_a] \Theta\eta_+\rangle \\ &\quad + \tilde{\nabla}^2 \|\eta_+\|^2 + \frac{1}{4} [(e^{-z} - 1)^2 + |x|^2] \tilde{\nabla}^2 \|\Theta\eta_+\|^2 \\ &\quad + \tilde{\nabla}^2 \operatorname{Re}\langle\eta_+, [(e^{-z} - 1)\Gamma_z + x^a \Gamma_a] \Theta\eta_+\rangle\end{aligned}\tag{B.10.16}$$

Equation (B.10.1) thus separates into components proportional to the linearly independent functions, 1, e^{-z} , e^{-2z} , x^a , and $|x|^2$. The $|x|^2$ component,

$$\frac{1}{4} \tilde{\nabla}^2 \|\Theta\eta_+\|^2 + \frac{3}{2A} \partial^i A \partial_i \|\Theta\eta_+\|^2 = \frac{1}{2} \left\| \nabla^{(+)}(\Theta\eta_+) \right\|^2 + \frac{1}{4} \left\| \mathcal{A}^{(+)} \Theta\eta_+ \right\|^2,\tag{B.10.17}$$

by the Hopf maximum principle, tells us that if \mathcal{S} is compact then $\|\Theta\eta_+\|^2$ must be constant. This also means tha

$$\nabla_{\underline{i}}^{(+)}(\Theta\eta_+) = 0\tag{B.10.18}$$

and

$$\mathcal{A}^{(+)} \Theta\eta_+ = 0,\tag{B.10.19}$$

which are precisely the mixed AdS-transverse integrability conditions.

Furthermore, we can apply these to simplify the remaining components. The right hand side reduces to two terms,

$$2 \left\| \nabla_8^{(+)} \eta_{8+} \right\|^2 + \left\| \mathcal{A}^{(+)} \eta_{8+} \right\|^2 = 2 \left\| \nabla^{(+)} \eta_+ \right\|^2 + \left\| \mathcal{A}^{(+)} \eta_+ \right\|^2,\tag{B.10.20}$$

and the left hand side reduces to

$$\tilde{\nabla}_8^i \tilde{\nabla}_{8i} \|\eta_{8+}\|^2 - h^i \tilde{\nabla}_{8i} \|\eta_{8+}\|^2 \quad (\text{B.10.21})$$

$$\begin{aligned} &= \frac{2e^{-z}}{A^2} \left(\|\Theta\eta_+\|^2 - 2\text{Re}\langle\eta_+, \Gamma_z \Theta\eta_+\rangle \right) \\ &\quad + \frac{6}{A} \partial^i A \partial_i \|\eta_+\|^2 + \tilde{\nabla}^2 \|\eta_+\|^2 \\ &\quad + \frac{6}{A} \partial^i A \partial_i \text{Re}\langle\eta_+, [(e^{-z} - 1)\Gamma_z + x^a \Gamma_a] \Theta\eta_+\rangle \\ &\quad + \tilde{\nabla}^2 \text{Re}\langle\eta_+, [(e^{-z} - 1)\Gamma_z + x^a \Gamma_a] \Theta\eta_+\rangle. \end{aligned} \quad (\text{B.10.22})$$

Note that this eliminates the e^{-2z} component as well.

The 1 component is then

$$\tilde{\nabla}^2 f(\eta_+) + \frac{6}{A} \partial^i A \partial_i f(\eta_+) = 2 \left\| \nabla^{(+)} \eta_+ \right\|^2 + \left\| \mathcal{A}^{(+)} \eta_+ \right\|^2, \quad (\text{B.10.23})$$

where

$$f(\eta_+) = \|\eta_+\|^2 - \text{Re}\langle\eta_+, \Gamma_z \Theta\eta_+\rangle. \quad (\text{B.10.24})$$

Once again applying the Hopf maximum principle, we find that the Killing spinor equations are satisfied on the transverse space, \mathcal{S} . Together with equations (B.10.18) and (B.10.19) and the assumption of integrability on the AdS space, this implies that ϵ_+ is a Killing spinor on the entire space $\text{AdS}_6 \times_w \mathcal{S}_4$.

B.10.2 Lichnerowicz Theorem on η_-

For 8 compact dimensions, we know that

$$\int_{\mathcal{S}_8} \left\| \mathcal{D}_8^{(-)} \eta_- \right\|^2 = \int_{\mathcal{S}_8} \left\| \nabla_8^{(-)} \eta_{8-} \right\|^2 + \frac{1}{2} \int_{\mathcal{S}_8} \left\| \mathcal{A}^{(-)} \eta_{8-} \right\|^2 + \int_{\mathcal{S}_8} \text{Re}\langle \mathcal{B} \eta_{8-}, \mathcal{D}_8^{(-)} \eta_{8-} \rangle. \quad (\text{B.10.25})$$

When not all 8 dimensions are compact, however, there are additional surface terms in this equation which are non-zero. In this case, the condition on the integrands is (??),

$$\begin{aligned} \left\| \mathcal{D}_8^{(-)} \eta_{8-} \right\|^2 &= \left\| \nabla_8^{(-)} \eta_{8-} \right\|^2 + \frac{1}{2} \left\| \mathcal{A}^{(-)} \eta_{8-} \right\|^2 - \text{Re}\langle \eta_{8-}, \mathcal{B} \mathcal{D}_8^{(-)} \eta_{8-} \rangle \\ &\quad + \tilde{\nabla}_{8\bar{i}} \text{Re}\langle \eta_{8-}, \mathcal{F}_8^i \eta_{8-} \rangle, \end{aligned} \quad (\text{B.10.26})$$

where

$$\mathcal{B} = \left(-\frac{1}{2} h_{\underline{j}} \Gamma^j + \frac{i}{6} Y_{\underline{\ell}_1 \underline{\ell}_2 \underline{\ell}_3} \Gamma^{\ell_1 \ell_2 \ell_3} \right) + \left(\frac{3}{8} \Phi_{\underline{j}} \Gamma^j + \frac{1}{48} H_{\underline{\ell}_1 \underline{\ell}_2 \underline{\ell}_3} \Gamma^{\ell_1 \ell_2 \ell_3} \right) C * \quad (\text{B.10.27})$$

and

$$\begin{aligned} \mathcal{F}_8^i &= \Gamma^{ij} \nabla_{\underline{j}}^{(-)} + \left(-\frac{1}{4} h^i + \frac{i}{4} Y_{\underline{\ell}_1 \underline{\ell}_2}^i \Gamma^{\ell_1 \ell_2} \right) \\ &\quad + \left(\frac{17}{16} \Phi^i - \frac{11}{96} H_{\underline{\ell}_1 \underline{\ell}_2 \underline{\ell}_3}^i \Gamma^{i \ell_1 \ell_2 \ell_3} \right) C *. \end{aligned} \quad (\text{B.10.28})$$

For the AdS₆ case specifically, we will express the 8-dimensional quantities in terms of their AdS and transverse components as

$$\mathcal{D}_8^{(-)} = \Gamma^z \nabla_{8\bar{z}}^{(-)} + \Gamma^a \nabla_{8\bar{a}}^{(-)} + \mathcal{D}^{(-)} \quad (\text{B.10.29})$$

$$h = -\frac{2}{A}e^z - \frac{2}{A}dA \quad (\text{B.10.30})$$

$$\nabla_{8\bar{z}}^{(-)}\psi = \partial_{\bar{z}}\psi + \frac{1}{2A}(\Gamma_z\Theta - 2)\psi \quad (\text{B.10.31})$$

$$\nabla_{8\bar{a}}^{(-)}\psi = \partial_{\bar{a}}\psi + \frac{1}{2A}\Gamma_a\Theta\psi \quad (\text{B.10.32})$$

$$\mathcal{A}^{(-)} = \bar{P}_{\bar{i}}\Gamma^i + \frac{1}{4}\bar{X}_{\bar{i}}\Gamma^i{}_* \quad (\text{B.10.33})$$

and

$$\begin{aligned} \eta_{8-} &= \epsilon_{-}(0, 0, z, x^a, y^i), \\ &= e^z \left[1 + \left(\frac{e^{-z} - 1}{2} \Gamma_z - \frac{x^a}{2} \Gamma_a \right) \Theta \right] \eta_{-}(y^i). \end{aligned} \quad (\text{B.10.34})$$

The norm-squared of η_{8-} is

$$\begin{aligned} \|\eta_{8-}\|^2 &= e^{2z} \|\eta_{-}\|^2 + \frac{e^{2z}}{4} \left[(e^{-z} - 1)^2 + |x|^2 \right] \|\Theta\eta_{-}\|^2 \\ &\quad + 2e^{2z} \text{Re} \left\langle \eta_{-}, \left(\frac{e^{-z} - 1}{2} \Gamma_z - \frac{x^a}{2} \Gamma_a \right) \Theta \eta_{-} \right\rangle. \end{aligned} \quad (\text{B.10.35})$$

If we assume that $\mathcal{D}^{(-)}\epsilon_{-} = \mathcal{D}^{(-)}(\Theta\epsilon_{-}) = \Theta^2\epsilon_{-} = 0$, we see immediately that $\nabla_{8\bar{z}}^{(-)}\eta_{8-} = \nabla_{8\bar{a}}^{(-)}\eta_{8-} = 0$, so that $\mathcal{D}_8^{(-)}\eta_{8-} = \mathcal{D}^{(-)}\eta_{-} = 0$. The other terms in equation (B.10.26) can be

expanded as

$$\left\| \nabla_8^{(-)} \eta_{8-} \right\|^2 = e^{2z} \left\| \nabla^{(-)} \eta_- + \left(\frac{e^{-z}-1}{2} \Gamma_z - \frac{x^a}{2} \Gamma_a \right) \nabla^{(-)} (\Theta \eta_-) \right\|^2 \quad (\text{B.10.36})$$

$$\begin{aligned} &= e^{2z} \left\| \nabla^{(-)} \eta_- \right\|^2 + \frac{e^{2z}}{4} \left[(e^{-z}-1)^2 + |x|^2 \right] \left\| \nabla^{(-)} (\Theta \eta_-) \right\|^2 \\ &\quad + 2e^{2z} \text{Re} \left\langle \nabla_i^{(-)} \eta_-, \left(\frac{e^{-z}-1}{2} \Gamma_z - \frac{x^a}{2} \Gamma_a \right) \nabla^{(-)i} (\Theta \eta_-) \right\rangle \end{aligned} \quad (\text{B.10.37})$$

$$\left\| \mathcal{A}^{(-)} \eta_{8-} \right\|^2 = e^{2z} \left\| \mathcal{A}^{(-)} \eta_- - \left(\frac{e^{-z}-1}{2} \Gamma_z - \frac{x^a}{2} \Gamma_a \right) \mathcal{A}^{(-)} \Theta \eta_- \right\|^2 \quad (\text{B.10.38})$$

$$\begin{aligned} &= e^{2z} \left\| \mathcal{A}^{(-)} \eta_- \right\|^2 + \frac{e^{2z}}{4} \left[(e^{-z}-1)^2 + |x|^2 \right] \left\| \mathcal{A}^{(-)} \Theta \eta_- \right\|^2 \\ &\quad - 2e^{2z} \text{Re} \left\langle \mathcal{A}^{(-)} \eta_-, \left(\frac{e^{-z}-1}{2} \Gamma_z - \frac{x^a}{2} \Gamma_a \right) \mathcal{A}^{(-)} \Theta \eta_- \right\rangle \end{aligned} \quad (\text{B.10.39})$$

$$\begin{aligned} &\tilde{\nabla}_{8\bar{i}} \text{Re} \left\langle \eta_{8-}, \mathcal{F}_{\bar{8}}^i \eta_{8-} \right\rangle \\ &= \partial_{\bar{z}} \text{Re} \left\langle \eta_{8-}, \left(\Gamma^z \mathcal{D}^{(-)} + \frac{1}{2A} - \frac{11}{96} G_{\bar{i}\bar{j}\bar{k}} \Gamma^{zijk} C^* \right) \eta_{8-} \right\rangle \end{aligned} \quad (\text{B.10.40})$$

$$\begin{aligned} &\quad + \partial_{\bar{a}} \text{Re} \left\langle \eta_{8-}, \left(\Gamma^a \mathcal{D}^{(-)} - \frac{11}{96} G_{\bar{i}\bar{j}\bar{k}} \Gamma^{aijk} C^* \right) \eta_{8-} \right\rangle \\ &\quad + \tilde{\nabla}_{\bar{i}} \text{Re} \left\langle \eta_{8-}, \mathcal{F}^i \eta_{8-} \right\rangle + \frac{4}{A} \partial_{\bar{i}} A \text{Re} \left\langle \eta_{8-}, \mathcal{F}^i \eta_{8-} \right\rangle \\ &\quad + \frac{3}{A} \text{Re} \left\langle \eta_{8-}, \left(\Gamma^z \mathcal{D}^{(-)} + \frac{1}{2A} - \frac{11}{96} G_{\bar{i}\bar{j}\bar{k}} \Gamma^{zijk} C^* \right) \eta_{8-} \right\rangle \\ &= \frac{1}{2A^2} \partial_{\bar{z}} \|\eta_{8-}\|^2 + \frac{3}{2A^2} \|\eta_{8-}\|^2 + \tilde{\nabla}_{\bar{i}} \text{Re} \left\langle \eta_{8-}, \mathcal{F}^i \eta_{8-} \right\rangle \end{aligned} \quad (\text{B.10.41})$$

$$\begin{aligned} &\quad - \frac{1}{A} \partial_{\bar{z}} \text{Re} \left\langle \eta_{8-}, \frac{11}{96} X_{\bar{i}} \Gamma^{zi} * \eta_{8-} \right\rangle + \frac{4}{A} \partial_{\bar{i}} A \text{Re} \left\langle \eta_{8-}, \mathcal{F}^i \eta_{8-} \right\rangle \\ &\quad - \frac{e^{-z}}{A} \partial_a \text{Re} \left\langle \eta_{8-}, \frac{11}{96} X_{\bar{i}} \Gamma^{ai} * \eta_{8-} \right\rangle - \frac{3}{A} \text{Re} \left\langle \eta_{8-}, \frac{11}{96} X_{\bar{i}} \Gamma^{zi} * \eta_{8-} \right\rangle \\ &= \frac{5}{2A^2} \|\eta_{8-}\|^2 + \frac{1}{4A^2} (e^z - 1) \|\Theta \eta_-\|^2 - \frac{1}{2A^2} e^z \text{Re} \left\langle \eta_-, \Gamma_z \Theta \eta_- \right\rangle \\ &\quad + \tilde{\nabla}_{\bar{i}} \text{Re} \left\langle \eta_{8-}, \mathcal{F}^i \eta_{8-} \right\rangle - \frac{5}{A} \text{Re} \left\langle \eta_{8-}, \frac{11}{96} X_{\bar{i}} \Gamma^{zi} * \eta_{8-} \right\rangle \end{aligned} \quad (\text{B.10.42})$$

where we've used the fact that $\Theta^2 \eta_- = 0$ to find that

$$\begin{aligned} \text{Re} \left\langle \eta_{8-}, \frac{11}{96} X_{\bar{i}} \Gamma^{zi} * \eta_{8-} \right\rangle &= \frac{11}{12A} e^{2z} \left(\|\eta_-\|^2 + \frac{1}{4} \left[1 - e^{-2z} + |x|^2 \right] \|\Theta \eta_-\|^2 \right. \\ &\quad \left. + 2 \text{Re} \left\langle \eta_-, \left(\frac{e^{-z}-1}{2} \Gamma_z - \frac{x^a}{2} \Gamma_a \right) \Theta \eta_- \right\rangle \right) \\ &\quad - \frac{11}{12A} e^z \text{Re} \left\langle \eta_-, \Gamma^z \Theta \eta_- \right\rangle \end{aligned} \quad (\text{B.10.43})$$

$$\text{Re} \left\langle \eta_{8-}, \frac{11}{96} X_{\bar{i}} \Gamma^{ai} * \eta_{8-} \right\rangle = \frac{11}{12A} e^z \left(\frac{x^a}{2} \|\Theta \eta_-\|^2 - \text{Re} \left\langle \eta_-, \Gamma^a \Theta \eta_- \right\rangle \right) \quad (\text{B.10.44})$$

B.11 Homogeneity implies constant warp factor

The computations for flat backgrounds are the same as those for AdS backgrounds, except that $\ell = \infty$, i.e. $\frac{1}{\ell} = 0$. Specifically, the $\mathbb{R}^{n-1,1} \times_w M^{10-n}$ metric is

$$ds^2 = 2du(dr + rh) + A^2 \left(\sum_{a=2}^{n-1} (dx^a)^2 \right) + ds^2(M^{10-n}) , \quad (\text{B.11.1})$$

$$h = -2A^{-1}dA . \quad (\text{B.11.2})$$

Using the obvious frame,

$$e^+ = du, \quad e^- = dr - 2rA^{-1}dA, \quad e^a = Adx^a, \quad e^i = e_M^i, \quad (\text{B.11.3})$$

the frame-indexed derivatives are

$$\partial_+ = \partial_u, \quad \partial_- = \partial_r, \quad \partial_a = A^{-1}\partial_{x^a}, \quad \partial_i = (e_M)^i_j (\partial_{y^j} + 2rA^{-1}\partial_{y^j}A \partial_r) , \quad (\text{B.11.4})$$

and the spin connection is

$$\Omega^\pm_\pm = \pm A^{-1}dA, \quad \Omega^\mu_i = A^{-1}\partial_i A e^\mu, \quad \Omega^i_j = \Omega_M^i_j. \quad (\text{B.11.5})$$

Using this spin connection in the flat components of the gravitino KSE, they can all be expressed as

$$\partial_\mu \epsilon_\pm + \Gamma_\mu \Theta_\pm \epsilon_\pm = 0 , \quad (\text{B.11.6})$$

where Θ_\pm for $\mathbb{R}^{n-1,1}$ is the same as Θ_\pm for AdS_n with $\frac{1}{\ell} = 0$. The integrability condition is then $\Theta_\mp \Theta_\pm \epsilon_\pm = 0$, or, defining $\Xi_\pm = \Gamma_2 \Theta_\pm$, $\Xi_\pm^2 \epsilon_\pm = 0$.

The homogeneity conjecture tells us that the Killing vectors derived from the Killing spinors, $K^M = \langle \epsilon, \Gamma^M \epsilon \rangle$, span the tangent space at each point. Using the flat-flat part of the Killing vector condition,

$$0 = \nabla_{(\mu} K_{\nu)} = \partial_{(\mu} K_{\nu)} - \frac{1}{A} K^i \partial_i A \eta_{\mu\nu} , \quad (\text{B.11.7})$$

because $\partial_\mu K_\nu = \partial_\mu \langle \sigma, \Gamma_\nu \sigma \rangle = 0$, $\mathcal{L}_K A = 0$ for each of these Killing vectors. Then, because they span the tangent space, we find that A must be constant over the entire space, and, in particular, over the transverse space.

B.12 IIB AdS₃

B.12.1 Chirality

For any IIB spinors, ψ ,

$$\Gamma_{0123456789} \psi = \psi. \quad (\text{B.12.1})$$

For ψ_\pm defined such that $\Gamma_\pm \psi_\pm = 0$, this means that

$$\Gamma_{23456789} \psi_\pm = \pm \psi_\pm, \quad (\text{B.12.2})$$

and

$$\Gamma_{3456789} \psi_\pm = \pm \Gamma_2 \psi_\pm. \quad (\text{B.12.3})$$

B.12.2 Electric and Magnetic Components of Fields

For $k \leq 5$, the fields will have AdS components, and so we must consider how they break down into AdS and transverse components. For $k = 3$, Q and P are purely transverse, while G and F have both AdS and transverse components. G can be expressed in terms of a scalar, Φ , and a transverse three-form, H ,

$$G = \Phi d\text{vol}(AdS_3) + H, \quad (\text{B.12.4})$$

while F can be expressed in terms of a two-form, Y ,

$$F = d\text{vol}(AdS_3) \wedge Y - \star_7 Y. \quad (\text{B.12.5})$$

In index notation these are

$$G_{+-z} = \Phi \quad (\text{B.12.6})$$

$$G_{ijk} = H_{ijk} \quad (\text{B.12.7})$$

$$F_{+-zij} = Y_{ij} \quad (\text{B.12.8})$$

$$F_{\underline{k_1 k_2 k_3 k_4 k_5}} = -\frac{1}{2} \epsilon^{ij}{}_{k_1 k_2 k_3 k_4 k_5} Y_{ij}. \quad (\text{B.12.9})$$

B.12.3 Field Equations and Bianchi Identities

The Einstein equation has an AdS component,

$$\frac{1}{A} \tilde{\nabla}^2 A = 2Y^2 + \frac{3}{8} |\Phi|^2 + \frac{1}{48} |H|^2 - \frac{2}{A^2} - \frac{2}{A^2} |dA|^2, \quad (\text{B.12.10})$$

and a transverse component

$$\begin{aligned} 0 = \tilde{R}_{ij} - \frac{3}{A} \tilde{\nabla}_i \tilde{\nabla}_j A - 2Y^2 \delta_{ij} + 8Y_{ik} Y_j{}^k \\ - \frac{1}{4} H_{(i}{}^{k\ell} \overline{H}_{j)k\ell} - \frac{1}{8} |\Phi|^2 \delta_{ij} + \frac{1}{48} |H|^2 \delta_{ij} - 2|P|^2. \end{aligned} \quad (\text{B.12.11})$$

Contracting this, we find that

$$\tilde{R} = \frac{3}{A} \tilde{\nabla}^2 A + 6Y^2 + \frac{5}{48} |H|^2 + \frac{7}{8} |\Phi|^2 + 2|P|^2 \quad (\text{B.12.12})$$

$$= -\frac{6}{A^2} - \frac{6}{A} |dA|^2 + 12Y^2 + 2|\Phi|^2 + \frac{1}{6} |H|^2 + 2|P|^2. \quad (\text{B.12.13})$$

The Bianchi identities reduce to

$$dY = -\frac{3}{A} dA \wedge Y + \frac{i}{8} (\overline{\Phi} H - \Phi \overline{H}) \quad (\text{B.12.14})$$

$$d \star_7 Y = -\frac{i}{8} H \wedge \overline{H} \quad (\text{B.12.15})$$

$$d\Phi = \frac{3}{A} \Phi dA + i\Phi Q - \overline{\Phi} P \quad (\text{B.12.16})$$

$$dH = iQ \wedge H - P \wedge \overline{H} \quad (\text{B.12.17})$$

$$dP = 2iQ \wedge P \quad (\text{B.12.18})$$

$$dQ = -iP \wedge \overline{P}, \quad (\text{B.12.19})$$

while the field equations reduce to

B.12.4 Parallel Transport Equations

In the AdS directions the parallel transport equations are

$$\partial_u \epsilon + \frac{1}{2A} \Gamma_+ \Theta_- \epsilon = 0 \quad (\text{B.12.20})$$

$$\partial_r \epsilon + \frac{1}{2A} \Gamma_- \Theta_+ \epsilon = 0 \quad (\text{B.12.21})$$

$$\partial_z \epsilon_{\pm} - \Xi_{\pm} \epsilon_{\pm} = 0 \quad (\text{B.12.22})$$

where

$$\Theta_{\pm} = \Gamma_z + \partial^i A \Gamma_i \mp \frac{iA}{2} Y^{ij} \Gamma_{zij} + \left(-\frac{2A}{96} H^{ijk} \Gamma_{ijk} \pm \frac{36A}{96} \Phi \Gamma_z \right) C* \quad (\text{B.12.23})$$

$$\Xi_+ = -\frac{1}{2} \Gamma_z \Theta_+, \quad \Xi_- = 1 - \frac{1}{2} \Gamma_z \Theta_-. \quad (\text{B.12.24})$$

The AdS integrability condition is $\Theta_{\mp} \Theta_{\pm} \epsilon_{\pm} = 0$, which implies that

$$\epsilon_{\pm}(0, 0, z, 0, y^i) = \sigma_{\pm}(y^i) + e^{\mp z} \tau_{\pm}(y^i). \quad (\text{B.12.25})$$

Introducing

$$\tilde{\Xi}_{\pm} = \mp \frac{c}{2} + \frac{1}{2} \not{A} \Gamma_z \pm \frac{iA}{4} \not{Y} + \left(-\frac{A}{96} \not{H} \Gamma^z \mp \frac{18A}{96} \Phi \right) C*, \quad (\text{B.12.26})$$

the conditions on σ_{\pm} and τ_{\pm} are

$$\tilde{\Xi}_{\pm} \chi_{\pm} = 0 \quad (\text{B.12.27})$$

where $c = 1$ when $\chi_{\pm} = \sigma_{\pm}$ and $c = -1$ when $\chi_{\pm} = \tau_{\pm}$.

The parallel transport equations in the transverse dimensions are

$$0 = \nabla_i^{(\pm)} \epsilon \quad (\text{B.12.28})$$

$$= \tilde{\nabla}_i \epsilon + \Psi_i^{(\pm)} \epsilon_{\pm} \quad (\text{B.12.29})$$

where

$$\begin{aligned} \Psi_i^{(\pm)} = & \pm \frac{1}{2A} \partial_i A - \frac{i}{2} Q_i \pm \frac{i}{4} (\Gamma \not{Y})_i \Gamma_z \mp \frac{i}{2} \not{Y}_i \Gamma^z \\ & + \left(-\frac{1}{96} (\Gamma \not{H})_i + \frac{9}{96} \not{H}_i \mp \frac{6}{96} \Phi \Gamma_{zi} \right) C*. \end{aligned} \quad (\text{B.12.30})$$

B.12.5 Maximality Condition on σ_+ and τ_+

We introduce a new operator,

$$\hat{\nabla}_i^{(+)} = \nabla_i^{(+)} + \frac{q}{A} \Gamma_{zi} \tilde{\Xi}_+, \quad (\text{B.12.31})$$

with the intention to demonstrate that, for an appropriately chosen value of q , if $\Gamma^i \hat{\nabla}_i^{(+)} \chi_+ = 0$, then χ_+ satisfies the Killing spinor equations.

As before, we find that

$$\begin{aligned}
\tilde{\nabla}^2 \|\chi_+\|^2 &= 2 \left\| \hat{\nabla}^{(+)} \chi_+ \right\|^2 + \frac{1}{2} \tilde{R} \|\chi_+\|^2 \\
&+ \text{Re} \left\langle \chi_+, \left[-4\bar{\Psi}^{(+i)} - 2\Psi^{(+i)} - 2\Gamma^{ij}\Psi_j^{(+)} \right. \right. \\
&\quad \left. \left. - 14\frac{q}{A}\Gamma^{zi}\tilde{\Xi}_+ \right] \tilde{\nabla}_i \chi_+ \right\rangle \\
&= \text{Re} \left\langle \chi_+, \left[-2\left(\bar{\Psi}^{(+i)} + \frac{q}{A}\tilde{\Xi}_+ \Gamma^{zi} \right) \left(\Psi_i^{(+)} + \frac{q}{A}\Gamma_{zi}\tilde{\Xi}_+ \right) \right. \right. \\
&\quad \left. \left. - 2\tilde{\nabla}^i \Psi_i^{(+)} - 2\Gamma^{ij}\tilde{\nabla}_i \Psi_j^{(+)} - 14\tilde{\nabla}_i \left(\frac{q}{A}\Gamma^{zi}\tilde{\Xi}_+ \right) \right] \chi_+ \right\rangle,
\end{aligned} \tag{B.12.32}$$

where

$$\begin{aligned}
\bar{\Psi}_i^{(+)} &= \frac{1}{2A} \partial_i A + \frac{i}{2} Q_i - \frac{i}{4} (\Gamma \mathcal{Y})_i \Gamma_z - \frac{i}{2} \mathcal{Y}_i \Gamma_z \\
&+ \left(-\frac{1}{96} (\Gamma \mathcal{H})_i - \frac{9}{96} \mathcal{H}_i + \frac{6}{96} \Phi \Gamma_{zi} \right) C^* \\
\tilde{\Xi}_+ &= -\frac{c}{2} - \frac{1}{2} \not\partial A \Gamma_z + \frac{iA}{4} \mathcal{Y} + \left(-\frac{A}{96} \mathcal{H} \Gamma_z - \frac{18A}{96} \Phi \right) C^*.
\end{aligned} \tag{B.12.33}$$

$$\tag{B.12.34}$$

Expanding the third term, we find that

$$\begin{aligned}
&\text{Re} \left\langle \chi_+, \left[-4\bar{\Psi}^{(+i)} - 2\Psi^{(+i)} - 2\Gamma^{ij}\Psi_j^{(+)} - 14\frac{q}{A}\Gamma^{zi}\tilde{\Xi}_+ \right] \tilde{\nabla}_i \chi_+ \right\rangle \\
&= \text{Re} \left\langle \chi_+, \left[\frac{7qc}{A} \Gamma^{zi} - \frac{3+7q}{A} \partial^i A - \frac{1+7q}{A} (\Gamma \not\partial A)^i \right. \right. \\
&\quad \left. \left. - iQ^i + i(\Gamma \mathcal{Q})^i + \frac{-2+14q}{2} i\mathcal{Y}^i \Gamma^z + \frac{-1+7q}{2} i(\Gamma \mathcal{Y})^i \Gamma^z \right. \right. \\
&\quad \left. \left. + \left(\frac{-6+14q}{96} (\Gamma \mathcal{H})^i + \frac{6+42q}{96} \mathcal{H}^i + \frac{60+252q}{96} \Phi \Gamma^{zi} \right) C^* \right] \tilde{\nabla}_i \chi_+ \right\rangle.
\end{aligned} \tag{B.12.35}$$

We would like to write this in the form

$$\alpha \partial^i A \tilde{\nabla}_i \|\chi_+\|^2 + \text{Re} \left\langle \chi_+, \mathcal{F} \Gamma^i \tilde{\nabla}_i \chi_+ \right\rangle, \tag{B.12.36}$$

which is only possible if $q = \frac{1}{7}$, in which case $\alpha = -\frac{3}{A}$,

$$\mathcal{F} = \frac{c}{A} \Gamma^z + \frac{2}{A} \not\partial A - i\mathcal{Q} + \left(\frac{1}{24} \mathcal{H} + \Phi \Gamma^z \right) C^*, \tag{B.12.37}$$

and

$$\begin{aligned}
&\text{Re} \left\langle \chi_+, \left[-4\bar{\Psi}^{(+i)} - 2\Psi^{(+i)} - 2\Gamma^{ij}\Psi_j^{(+)} - 12\frac{q}{A}\Gamma^{zi}\tilde{\Xi}_+ \right] \tilde{\nabla}_i \chi_+ \right\rangle \\
&= -\frac{4}{A} \partial^i A \tilde{\nabla}_i \|\chi_+\|^2 + \text{Re} \left\langle \chi_+, \mathcal{F} \Gamma^i \tilde{\nabla}_i \chi_+ \right\rangle
\end{aligned} \tag{B.12.38}$$

$$= -\frac{4}{A} \partial^i A \tilde{\nabla}_i \|\chi_+\|^2 - \text{Re} \left\langle \chi_+, \mathcal{F} \Gamma^i \left[\Psi_i^{(+)} + \frac{1}{7A} \Gamma_{zi} \tilde{\Xi}_+ \right] \chi_+ \right\rangle. \tag{B.12.39}$$

Combining this with the second term and the bilinear part of the fourth term in (B.14.32), we find that

$$\begin{aligned} & \text{Re} \left\langle \chi_+, -2 \left(\bar{\Psi}^{(+)} + \frac{1}{7A} \tilde{\Xi}_+ \Gamma^{zi} + \frac{1}{2} \mathcal{F} \Gamma^i \right) \left(\Psi_i^{(+)} + \frac{1}{7A} \Gamma_{zi} \tilde{\Xi}_+ \right) \chi_+ \right\rangle \\ &= \text{Re} \left\langle \chi_+, -2 \left[\frac{3c}{7A} \Gamma^{zi} + \frac{10}{7A} \partial^i A - \frac{13}{14A} (\Gamma \not{A})^i + \frac{i}{2} (\Gamma Q)^i - \frac{3i}{7} \not{Y}^i \Gamma_z \right. \right. \end{aligned} \quad (\text{B.12.40})$$

$$\begin{aligned} & \left. - \frac{2i}{7} (\Gamma \not{X})^i \Gamma^z + \left(-\frac{20}{7 \cdot 96} (\Gamma \not{H})^i - \frac{24}{7 \cdot 96} \not{H}^i + \frac{15}{28} \Phi \Gamma^{zi} \right) C* \right] \\ & \times \left[-\frac{c}{14A} \Gamma_{zi} + \frac{4}{7A} \partial_i A + \frac{1}{14A} (\Gamma \not{A})_i - \frac{i}{2} Q_i - \frac{4i}{7} \not{Y}_i \Gamma_z \right. \\ & \left. + \frac{3i}{14} (\Gamma \not{X})_i \Gamma_z + \left(-\frac{8}{7 \cdot 96} (\Gamma \not{H})_i + \frac{60}{7 \cdot 96} \not{H}_i - \frac{5}{56} \Phi \Gamma_{zi} \right) C* \right] \chi_+ \Big\rangle \\ &= \text{Re} \left\langle \chi_+, \left[-\frac{3}{7A^2} - \frac{17}{7A^2} |dA|^2 - 2Y^2 - \frac{1}{7} \not{Y}^2 - \frac{4ic}{7A} \not{Y} - \frac{3}{28} |\Phi|^2 + \frac{5}{168} \Phi \not{H} \Gamma_z \right. \right. \\ & \left. - \frac{11}{7 \cdot 288} \not{H} \not{H} + \frac{1}{32} \not{H}^i \not{H}_i \right. \\ & \left. + \left(\frac{i}{12} Q_i (\Gamma \not{H})^i + \frac{c}{42A} \not{H} \Gamma_z - \frac{1}{42A} \partial_i A (\Gamma \not{H})^i - \frac{3c}{7A} \Phi \right) C* \right] \chi_+ \Big\rangle \end{aligned} \quad (\text{B.12.41})$$

We can use the field equations and Bianci identities to rewrite the last line of (B.14.32),

$$\begin{aligned} & \text{Re} \left\langle \chi_+, \left[-2 \tilde{\nabla}^i \Psi_i^{(+)} - 2 \Gamma^{ij} \tilde{\nabla}_i \Psi_j^{(+)} - \tilde{\nabla}_i \left(\frac{4}{A} \Gamma^{zi} \tilde{\Xi}_+ \right) \right] \chi_+ \right\rangle \\ &= \text{Re} \left\langle \chi_+, \left(\frac{2}{A^2} |dA|^2 - \frac{2}{A} \tilde{\nabla}^2 A + \frac{i}{2} dQ - 2i \tilde{\nabla}^i \not{Y}_i \Gamma_z - \frac{1}{48} dH C* \right) \chi_+ \right\rangle \\ &= \text{Re} \left\langle \chi_+, \left[\frac{4}{A^2} + \frac{6}{A^2} |dA|^2 - 4Y^2 - \frac{3}{4} |\Phi|^2 + \frac{5}{168} \Phi \not{H} \Gamma_z + P_i \bar{P}_i \right. \right. \\ & \left. + \frac{1}{144} \not{H} \not{H} - \frac{1}{16} \not{H}^i \not{H}_i - \frac{1}{8} \not{H}^{ij} \not{H}_{ij} + \frac{1}{24} |H|^2 \right. \\ & \left. + \left(-\frac{i}{12} Q_i (\Gamma \not{H})^i + \frac{1}{12} P_i \overline{(\Gamma \not{H})}^i \right) C* \right] \chi_+ \Big\rangle. \end{aligned} \quad (\text{B.12.42})$$

The second, third, and fourth terms on the right side of equation (B.14.32) thus sum to

$$\begin{aligned} & \text{Re} \left\langle \chi_+, \left[\frac{4}{7A^2} + \frac{4}{7A^2} |dA|^2 - \frac{4ic}{7A} \not{Y} - \frac{1}{7} \not{Y}^2 + \frac{1}{7} |\Phi|^2 + \frac{1}{84} \Phi \not{H} \Gamma_z \right. \right. \\ & \left. + |P|^2 + P_i \bar{P}_j \Gamma^{ij} \right. \\ & \left. \frac{1}{7 \cdot 96} \not{H} \not{H} - \frac{1}{32} \not{H}^i \not{H}_i - \frac{1}{8} \not{H}^{ij} \not{H}_{ij} + \frac{1}{24} |H|^2 \right. \\ & \left. + \left(\frac{c}{42A} \not{H} \Gamma_z - \frac{1}{42A} \partial_i A (\Gamma \not{H})^i + \frac{1}{12} P_i \overline{(\Gamma \not{H})}^i - \frac{3c}{7A} \Phi \right) C* \right] \chi_+ \Big\rangle \end{aligned} \quad (\text{B.12.43})$$

Noting that

$$\left\| \tilde{\Xi}_+ \chi_+ \right\|^2 = \left\langle \chi_+, \tilde{\Xi}_+ \tilde{\Xi}_+ \chi_+ \right\rangle \quad (\text{B.12.44})$$

$$\begin{aligned} &= \left\langle \chi_+, \left[\frac{1}{4} + \frac{1}{4} |dA|^2 - \frac{iA}{2} \mathcal{Y}^i \partial_i A \Gamma_z - \frac{iAc}{4} \mathcal{Y} - \frac{A^2}{16} \mathcal{Y}^2 \right. \right. \\ &\quad \left. \left. - \frac{A^2}{96^2} \mathcal{H} \overline{\mathcal{H}} + \frac{9A^2}{256} |\Phi|^2 + \frac{A^2}{256} \Phi \overline{\mathcal{H}} \Gamma_z \right. \right. \\ &\quad \left. \left. + \left(\frac{Ac}{96} \mathcal{H} \Gamma^z - \frac{A}{96} \partial_i A (\Gamma \mathcal{H})^i + \frac{iA^2}{32} \mathcal{Y}_i \mathcal{H}^i \Gamma_z - \frac{3Ac}{16} \Phi \right) C^* \right] \chi_+ \right\rangle \end{aligned} \quad (\text{B.12.45})$$

$$\begin{aligned} \left\| \mathcal{A}^{(+)} \chi_+ \right\|^2 &= \text{Re} \left\langle \chi, \left[|P|^2 + P_i \overline{P}_j \Gamma^{ij} + \frac{1}{576} \mathcal{H} \overline{\mathcal{H}} + \frac{1}{16} |\Phi|^2 \right. \right. \\ &\quad \left. \left. + \frac{1}{48} \Phi \overline{\mathcal{H}} \Gamma_z + \frac{1}{12} P_i \overline{(\Gamma \mathcal{H})}^i C^* \right] \chi \right\rangle \end{aligned} \quad (\text{B.12.46})$$

we can now write equation (B.14.32) as

$$\tilde{\nabla}^2 \|\chi\|^2 + \frac{4}{A} \partial^i A \tilde{\nabla}_i \|\chi\|^2 = 2 \left\| \hat{\nabla}^{(+)} \chi \right\|^2 + \frac{16}{7A^2} \|\Xi_c \chi\|^2 + \|\mathcal{A} \chi\|^2 \quad (\text{B.12.47})$$

B.13 IIB AdS₄

B.13.1 Chirality

For any IIB spinors, ψ ,

$$\Gamma_{0123456789} \psi = \psi. \quad (\text{B.13.1})$$

For ψ_{\pm} defined such that $\Gamma_{\pm} \psi_{\pm} = 0$, this means that

$$\Gamma_{23456789} \psi_{\pm} = \pm \psi_{\pm}, \quad (\text{B.13.2})$$

and

$$\Gamma_{456789} \psi_{\pm} = \mp \Gamma_{23} \psi_{\pm}. \quad (\text{B.13.3})$$

B.13.2 Electric and Magnetic Components of Fields

For $k \leq 5$, the fields will have AdS components, and so we must consider how they break down into AdS and transverse components. For $k = 4$, Q, P, and G are all purely transverse, but F, the self-dual five form, includes both components, and can be expressed in terms of a one form, Y ,

$$F = \text{dvol}(AdS_4) \wedge Y + \star_6 Y, \quad (\text{B.13.4})$$

or, in index notation,

$$F_{+-\underline{z}3i} = -\frac{1}{5!} \epsilon_{ij_1 j_2 j_3 j_4 j_5} F^{\underline{j_1 j_2 j_3 j_4 j_5}} = Y_i. \quad (\text{B.13.5})$$

B.13.3 Field Equations and Bianchi Identities

The Einstein equation has an AdS component,

$$\frac{1}{A}\tilde{\nabla}^2 A = 4Y^2 + \frac{1}{48}|G|^2 - \frac{3}{A^2} - \frac{3}{A^2}|dA|^2, \quad (\text{B.13.6})$$

and a transverse component,

$$\begin{aligned} \tilde{R}_{ij} - \frac{4}{A}\tilde{\nabla}_i\tilde{\nabla}_j A - 4Y^2\delta_{ij} + 8Y_iY_j \\ - \frac{1}{4}G_{(i}{}^{k\ell}\overline{G}_{j)k\ell} + \frac{1}{48}|G|^2\delta_{ij} - 2P_{(i}\overline{P}_{j)} = 0. \end{aligned} \quad (\text{B.13.7})$$

Contracting this, we find that the scalar curvature of the transverse space is

$$\tilde{R} = \frac{4}{A}\tilde{\nabla}^2 A + 16Y^2 + \frac{1}{8}|G|^2 + 2|P|^2 \quad (\text{B.13.8})$$

$$= -\frac{12}{A^2} - \frac{12}{A^2}|dA|^2 + 32Y^2 + \frac{5}{24}|G|^2 + 2|P|^2 \quad (\text{B.13.9})$$

The Bianchi identities reduce to

$$dY = -\frac{4}{A}dA \wedge Y \quad (\text{B.13.10})$$

$$\tilde{\nabla}^i Y_i = -\frac{i}{288}\epsilon^{i_1 i_2 i_3 j_1 j_2 j_3} G_{\underline{i_1 i_2 i_3}} \overline{G}_{\underline{j_1 j_2 j_3}} \quad (\text{B.13.11})$$

$$dG = iQ \wedge G - P \wedge \overline{G} \quad (\text{B.13.12})$$

$$dP = 2iQ \wedge P \quad (\text{B.13.13})$$

$$dQ = -iP \wedge \overline{P}, \quad (\text{B.13.14})$$

while the field equations reduce to

$$\tilde{\nabla}^i G_{ijk} = iQ^i G_{ijk} + P^i \overline{G}_{ijk} \quad (\text{B.13.15})$$

$$\tilde{\nabla}^i P_i = 2iQ^i P_i - \frac{1}{24}G^2. \quad (\text{B.13.16})$$

B.13.4 Parallel Transport Equations

In the AdS directions, the parallel transport Killing spinor equations are

$$\partial_u \epsilon + \frac{1}{2A}\Gamma_+ \Theta_- \epsilon = 0 \quad (\text{B.13.17})$$

$$\partial_r \epsilon + \frac{1}{2A}\Gamma_- \Theta_+ \epsilon = 0 \quad (\text{B.13.18})$$

$$\partial_z \epsilon_+ + \frac{1}{2}\Gamma_z \Theta_+ \epsilon_+ = 0 \quad (\text{B.13.19})$$

$$\partial_z \epsilon_- + \left(\frac{1}{2}\Gamma_z \Theta_- - 1 \right) \epsilon_- = 0 \quad (\text{B.13.20})$$

$$\partial_a \epsilon_{\pm} + \frac{e^z}{2}\Gamma_a \Theta_{\pm} \epsilon_{\pm} = 0 \quad (\text{B.13.21})$$

where

$$\Theta_{\pm} = \Gamma_z + \not\partial A \mp iA \not{Y} \Gamma_{z3} - \frac{A}{48} \not{C} C^*, \quad (\text{B.13.22})$$

and, for the z -direction equations, we're considering $r = 0$. The integrability condition for these directions is

$$\Theta_{\mp} \Theta_{\pm} \epsilon_{\pm} = 0, \quad (\text{B.13.23})$$

and we can conclude from this that the solutions to the z -direction equations are

$$\epsilon_{\pm}(0, 0, z, 0, y^i) = \sigma_{\pm}(y^i) + e^{\mp z} \tau_{\pm}(y^i), \quad (\text{B.13.24})$$

where, defining $\Xi_+ = -\frac{1}{2}\Gamma_z \Theta_+$ and $\Xi_- = 1 - \frac{1}{2}\Gamma_z \Theta_-$,

$$\Xi_{\pm} \sigma_{\pm} = 0 \quad (\text{B.13.25})$$

$$\Xi_{\pm} \tau_{\pm} = \mp \tau_{\pm}. \quad (\text{B.13.26})$$

We can write these conditions succinctly as

$$\tilde{\Xi}_{\pm} \chi_{\pm} = 0 \quad (\text{B.13.27})$$

where χ_{\pm} is either σ_{\pm} or τ_{\pm} ,

$$\tilde{\Xi}_{\pm} = \mp \frac{c}{2} + \frac{1}{2} \not\partial A \Gamma_z \mp \frac{iA}{2} \not{Y} \Gamma_3 - \frac{A}{96} \not{C} \Gamma^z C^*, \quad (\text{B.13.28})$$

and c is 1 when $\chi_{\pm} = \sigma_{\pm}$, -1 when $\chi_{\pm} = \tau_{\pm}$.

In the transverse directions, the parallel transport equation is

$$0 = \nabla_i^{(\pm)} \epsilon_{\pm} \quad (\text{B.13.29})$$

$$= \tilde{\nabla}_i \epsilon_{\pm} + \Psi_i^{(\pm)} \epsilon_{\pm}, \quad (\text{B.13.30})$$

where

$$\Psi_i^{(\pm)} = \pm \frac{1}{2A} \partial_i A - \frac{i}{2} Q_i \mp \frac{i}{2} Y_i \Gamma_{z3} \pm \frac{i}{2} (\Gamma \not{X})_i \Gamma_{z3} + \left(-\frac{1}{96} (\Gamma G)_i + \frac{9}{96} \not{C}_i \right) C^*. \quad (\text{B.13.31})$$

We can see that this applies independently to σ_{\pm} and τ_{\pm} , so that in general $\nabla_i^{(\pm)} \chi_{\pm} = 0$.

B.13.5 Maximality Condition on σ_+ and τ_+

We introduce a new operator,

$$\hat{\nabla}_i^{(+)} = \nabla_i^{(+)} + \frac{q}{A} \Gamma_{zi} \tilde{\Xi}_+, \quad (\text{B.13.32})$$

with the intention to demonstrate that, for an appropriately chosen value of q , if $\Gamma^i \hat{\nabla}_i^{(+)} \chi_+ = 0$, then χ_+ satisfies the Killing spinor equations.

As before, we find that

$$\begin{aligned}
\tilde{\nabla}^2 \|\chi_+\|^2 &= 2 \left\| \hat{\nabla}^{(+)} \chi_+ \right\|^2 + \frac{1}{2} \tilde{R} \|\chi_+\|^2 \\
&+ \text{Re} \left\langle \chi_+, \left[-4 \bar{\Psi}^{(+i)} - 2 \Psi^{(+i)} - 2 \Gamma^{ij} \Psi_j^{(+)} \right. \right. \\
&\quad \left. \left. - 12 \frac{q}{A} \Gamma^{zi} \tilde{\Xi}_+ \right] \tilde{\nabla}_i \chi_+ \right\rangle \\
&+ \text{Re} \left\langle \chi_+, \left[-2 \left(\bar{\Psi}^{(+i)} + \frac{q}{A} \tilde{\Xi}_+ \Gamma^{zi} \right) \left(\Psi_i^{(+)} + \frac{q}{A} \Gamma_{zi} \tilde{\Xi}_+ \right) \right. \right. \\
&\quad \left. \left. - 2 \tilde{\nabla}^i \Psi_i^{(+)} - 2 \Gamma^{ij} \tilde{\nabla}_i \Psi_j^{(+)} - 12 \tilde{\nabla}_i \left(\frac{q}{A} \Gamma^{zi} \tilde{\Xi}_+ \right) \right] \chi_+ \right\rangle,
\end{aligned} \tag{B.13.33}$$

where

$$\bar{\Psi}_i^{(+)} = \frac{1}{2A} \partial_i A + \frac{i}{2} Q_i - \frac{i}{2} Y_i \Gamma_{z3} - \frac{i}{2} (\Gamma \dot{Y})_i \Gamma_{z3} + \left(-\frac{1}{96} (\Gamma G)_i - \frac{9}{96} \mathcal{G}_i \right) C* \tag{B.13.34}$$

$$\tilde{\Xi}_+ = -\frac{c}{2} - \frac{1}{2} \not{\partial}^i A \Gamma_z - \frac{iA}{2} \not{Y} \Gamma_3 - \frac{A}{96} \mathcal{G} \Gamma^z C*. \tag{B.13.35}$$

Expanding the third term, we find that

$$\begin{aligned}
&\text{Re} \left\langle \chi_+, \left[-4 \bar{\Psi}^{(+i)} - 2 \Psi^{(+i)} - 2 \Gamma^{ij} \Psi_j^{(+)} - 12 \frac{q}{A} \Gamma^{zi} \tilde{\Xi}_+ \right] \tilde{\nabla}_i \chi_+ \right\rangle \\
&= \text{Re} \left\langle \chi_+, \left[\frac{6qc}{A} \Gamma^{zi} - \frac{3+6q}{A} \partial^i A - \frac{1+6q}{A} (\Gamma \not{\partial} A)^i - iQ^i + i(\Gamma Q)^i \right. \right. \\
&\quad \left. \left. + i(-2+6q)Y^i \Gamma^{z3} + i(-2+6q)(\Gamma \dot{Y})^i \Gamma^{z3} \right. \right. \\
&\quad \left. \left. + \left(\frac{-8+12q}{96} (\Gamma G)^i + \frac{36q}{96} \mathcal{G}^i \right) C* \right] \tilde{\nabla}_i \chi_+ \right\rangle.
\end{aligned} \tag{B.13.36}$$

We would like to write this in the form

$$\alpha \partial^i A \tilde{\nabla}_i \|\chi_+\|^2 + \text{Re} \left\langle \chi_+, \mathcal{F} \tilde{\nabla} \chi_+ \right\rangle, \tag{B.13.37}$$

which is only possible if $q = \frac{1}{3}$, in which case $\alpha = -\frac{4}{A}$,

$$\mathcal{F} = \frac{2c}{A} \Gamma^z + \frac{3}{A} \not{\partial} A - iQ + \frac{1}{24} \mathcal{G} C*, \tag{B.13.38}$$

and

$$\begin{aligned}
&\text{Re} \left\langle \chi_+, \left[-4 \bar{\Psi}^{(+i)} - 2 \Psi^{(+i)} - 2 \Gamma^{ij} \Psi_j^{(+)} - 12 \frac{q}{A} \Gamma^{zi} \tilde{\Xi}_+ \right] \tilde{\nabla}_i \chi_+ \right\rangle \\
&= -\frac{4}{A} \partial^i A \tilde{\nabla}_i \|\chi_+\|^2 + \text{Re} \left\langle \chi_+, \mathcal{F} \Gamma^i \tilde{\nabla}_i \chi_+ \right\rangle
\end{aligned} \tag{B.13.39}$$

$$= -\frac{4}{A} \partial^i A \tilde{\nabla}_i \|\chi_+\|^2 - \text{Re} \left\langle \chi_+, \mathcal{F} \Gamma^i \left[\Psi_i^{(+)} + \frac{1}{3A} \Gamma_{zi} \tilde{\Xi}_+ \right] \chi_+ \right\rangle. \tag{B.13.40}$$

Combining this with the second term and the bilinear part of the fourth term in (B.14.32), we

find that

$$\begin{aligned}
& \text{Re} \left\langle \chi_+, -2 \left(\bar{\Psi}^{(+i)} + \frac{1}{3A} \tilde{\Xi}_+ \Gamma^{zi} + \frac{1}{2} \mathcal{F} \Gamma^i \right) \left(\Psi_i^{(+)} + \frac{1}{3A} \Gamma_{zi} \Xi_+ \right) \chi_+ \right\rangle \\
&= \text{Re} \left\langle \chi_+, -2 \left[\frac{5c}{6A} \Gamma^{zi} + \frac{11}{6A} \partial^i A - \frac{4}{3A} (\Gamma \not{\partial} A)^i + \frac{i}{2} (\Gamma Q)^i - \frac{i}{3} Y^i \Gamma_{z3} \right. \right. \\
&\quad \left. \left. - \frac{2i}{3} (\Gamma \not{Y})^i \Gamma^{z3} + \left(-\frac{1}{36} (\Gamma G)^i - \frac{1}{24} \not{G}^i \right) C^* \right] \right. \\
&\quad \left. \times \left[-\frac{c}{6A} \Gamma_{zi} + \frac{2}{3A} \partial_i A + \frac{1}{6A} (\Gamma \not{\partial} A)_i - \frac{i}{2} Q_i - \frac{2i}{3} Y_i \Gamma_{z3} \right. \right. \\
&\quad \left. \left. + \frac{i}{3} (\Gamma \not{Y})_i \Gamma_{z3} + \left(-\frac{1}{72} (\Gamma G)_i + \frac{1}{12} \not{G}_i \right) C^* \right] \chi_+ \right\rangle
\end{aligned} \tag{B.13.41}$$

$$\begin{aligned}
&= \text{Re} \left\langle \chi_+, \left[-\frac{5}{3A^2} - \frac{14}{3A^2} |dA|^2 - \frac{8}{3} Y^i \partial_i A \Gamma_{z3} + \frac{8ic}{3A} \not{Y} \Gamma_3 - \frac{8i}{3A} Y^2 \right. \right. \\
&\quad \left. \left. + \left(\frac{i}{12} Q_i (\Gamma G)^i + \frac{c}{18A} \not{G} \Gamma^z - \frac{1}{18A} \partial_i A (\Gamma G)^i \right) C^* \right] \chi_+ \right\rangle
\end{aligned} \tag{B.13.42}$$

We can use the field equations and Bianci identities to rewrite the last line of (B.14.32),

$$\begin{aligned}
& \text{Re} \left\langle \chi_+, \left[-2 \tilde{\nabla}^i \Psi_i^{(+)} - 2 \Gamma^{ij} \tilde{\nabla}_i \Psi_j^{(+)} - \tilde{\nabla}_i \left(\frac{4}{A} \Gamma^{zi} \tilde{\Xi}_+ \right) \right] \chi_+ \right\rangle \\
&= \text{Re} \left\langle \chi_+, \left(\frac{3}{A^2} |dA|^2 - \frac{3}{A} \tilde{\nabla}^2 A + \frac{i}{2} dQ - 2i \tilde{\nabla}^i Y_i \Gamma_{z3} - \frac{1}{48} dGC^* \right) \chi_+ \right\rangle \\
&= \text{Re} \left\langle \chi_+, \left[\frac{9}{A^2} + \frac{12}{A^2} |dA|^2 - 12Y^2 - \frac{1}{16} |G|^2 + P_i \bar{P}_j \right. \right. \\
&\quad \left. \left. + \frac{1}{144} G_{i_1 i_2 i_3} \bar{G}_{j_1 j_2 j_3} \Gamma^{i_1 i_2 i_3 j_1 j_2 j_3} \right. \right. \\
&\quad \left. \left. + \left(-\frac{i}{12} Q_i (\Gamma G)^i + \frac{1}{12} P_i \overline{(\Gamma G)^i} \right) C^* \right] \chi_+ \right\rangle.
\end{aligned} \tag{B.13.43}$$

The second, third, and fourth terms on the right side of equation (B.14.32) thus sum to

$$\begin{aligned}
& \text{Re} \left\langle \chi_+, \left[\frac{4}{3A^2} + \frac{4}{3A^2} |dA|^2 - \frac{8i}{3} Y^i \partial_i A \Gamma_{z3} + \frac{8ic}{3A} \not{Y} \Gamma_3 + \frac{4}{3} Y^2 \right. \right. \\
&\quad \left. \left. + \frac{1}{24} |G|^2 + |P|^2 + P_i \bar{P}_j \Gamma^{ij} + \frac{1}{144} G_{i_1 i_2 i_3} \bar{G}_{j_1 j_2 j_3} \Gamma^{i_1 i_2 i_3 j_1 j_2 j_3} \right. \right. \\
&\quad \left. \left. + \left(-\frac{1}{36} (\Gamma G)^i - \frac{1}{24} \not{G}^i \right) \left(-\frac{1}{72} \overline{(\Gamma G)_i} + \frac{1}{12} \not{G}_i \right) \right. \right. \\
&\quad \left. \left. + \left(\frac{c}{18A} \not{G} \Gamma^z - \frac{1}{18A} \partial_i A (\Gamma G)^i + \frac{1}{12} P_i \overline{(\Gamma G)^i} \right) C^* \right] \chi_+ \right\rangle
\end{aligned} \tag{B.13.44}$$

Noting that

$$\left\| \tilde{\Xi}_+ \chi_+ \right\|^2 = \left\langle \chi_+, \tilde{\Xi}_+ \tilde{\Xi}_+ \chi_+ \right\rangle \quad (\text{B.13.45})$$

$$= \left\langle \chi_+, \left[\frac{1}{4} + \frac{1}{4} |dA|^2 + \frac{iA}{2} Y^i \partial_i A \Gamma_{z3} + \frac{iAc}{2} \not{Y} \Gamma_3 + \frac{A^2}{4} Y^2 - \frac{A^2}{96^2} \not{G} \not{G} + \left(\frac{Ac}{96} \not{G} \Gamma^z - \frac{A}{96} \partial_i A (\Gamma G)^i + \frac{iA^2}{32} Y^i \not{G}_i \Gamma^{z3} \right) C^* \right] \chi_+ \right\rangle \quad (\text{B.13.46})$$

$$\| \mathcal{A} \chi_+ \|^2 = \text{Re} \left\langle \chi, \left[|P|^2 + P_i \bar{P}_j \Gamma^{ij} - \frac{1}{576} \not{G} \not{G} + \frac{1}{12} P_i (\overline{\Gamma G})^i C^* \right] \chi \right\rangle \quad (\text{B.13.47})$$

we can now write equation (B.14.32) as

$$\tilde{\nabla}^2 \|\chi\|^2 + \frac{4}{A} \partial^i A \tilde{\nabla}_i \|\chi\|^2 = 2 \left\| \hat{\nabla}^{(+)} \chi \right\|^2 + \frac{16}{3A^2} \|\Xi_c \chi\|^2 + \|\mathcal{A} \chi\|^2 \quad (\text{B.13.48})$$

B.14 IIB AdS₅

B.14.1 Chirality

For any IIB spinors, ψ ,

$$\Gamma_{0123456789} \psi = \psi. \quad (\text{B.14.1})$$

For ψ_{\pm} defined such that $\Gamma_{\pm} \psi_{\pm} = 0$, this means that

$$\Gamma_{23456789} \psi_{\pm} = \pm \psi_{\pm}, \quad (\text{B.14.2})$$

and

$$\Gamma_{56789} \psi_{\pm} = \mp \Gamma_{234} \psi_{\pm}. \quad (\text{B.14.3})$$

B.14.2 Electric and Magnetic Components of Fields

For $k \leq 5$, the fields will have AdS components, and so we must consider how they break down into AdS and transverse components. For $k = 5$, Q, P, and G are all purely transverse, but F, the self-dual five form, includes both components, and can be expressed in terms of a scalar, Y ,

$$F = Y [\text{dvol}(AdS_5) + \text{dvol}(\mathcal{M}_5)], \quad (\text{B.14.4})$$

or, in index notation,

$$F_{+-\underline{z}34} = -F_{\underline{5}6789} = Y. \quad (\text{B.14.5})$$

B.14.3 Field Equations and Bianchi Identities

The Einstein equation has an AdS component,

$$\frac{1}{A} \tilde{\nabla}^2 A = 4Y^2 + \frac{1}{48} |G|^2 - \frac{4}{A^2} - \frac{4}{A^2} |dA|^2, \quad (\text{B.14.6})$$

and a transverse component,

$$\begin{aligned} \tilde{R}_{ij} - \frac{5}{A} \tilde{\nabla}_i \tilde{\nabla}_j A - 4Y^2 \delta_{ij} \\ - \frac{1}{4} G_{(i}{}^{k\ell} \overline{G}_{j)k\ell} + \frac{1}{48} |G|^2 \delta_{ij} - 2P_{(i} \overline{P}_{j)} = 0. \end{aligned} \quad (\text{B.14.7})$$

Contracting this, we find that the scalar curvature of the transverse space is

$$\tilde{R} = \frac{5}{A} \tilde{\nabla}^2 A + 20Y^2 + \frac{7}{48} |G|^2 + 2|P|^2 \quad (\text{B.14.8})$$

$$= -\frac{20}{A^2} - \frac{20}{A^2} |dA|^2 + 40Y^2 + \frac{1}{4} |G|^2 + 2|P|^2 \quad (\text{B.14.9})$$

The Bianchi identities reduce to

$$dY = -\frac{5}{A} Y dA \quad (\text{B.14.10})$$

$$dG = iQ \wedge G - P \wedge \overline{G} \quad (\text{B.14.11})$$

$$dP = 2iQ \wedge P \quad (\text{B.14.12})$$

$$dQ = -iP \wedge \overline{P}, \quad (\text{B.14.13})$$

while the field equations reduce to

$$\tilde{\nabla}^i G_{ijk} = -\frac{5}{A} \partial^i A G_{ijk} + iQ^i G_{ijk} + P^i \overline{G}_{ijk} \quad (\text{B.14.14})$$

$$\tilde{\nabla}^i P_i = -\frac{5}{A} \partial^i A P_i + 2iQ^i P_i - \frac{1}{24} G^2. \quad (\text{B.14.15})$$

B.14.4 Parallel Transport Equations

In the AdS directions, the parallel transport Killing spinor equations are

$$\partial_u \epsilon + \frac{1}{2A} \Gamma_+ \Theta_- \epsilon = 0 \quad (\text{B.14.16})$$

$$\partial_r \epsilon + \frac{1}{2A} \Gamma_- \Theta_+ \epsilon = 0 \quad (\text{B.14.17})$$

$$\partial_z \epsilon_+ + \frac{1}{2} \Gamma_z \Theta_+ \epsilon_+ = 0 \quad (\text{B.14.18})$$

$$\partial_z \epsilon_- + \left(\frac{1}{2} \Gamma_z \Theta_- - 1 \right) \epsilon_- = 0 \quad (\text{B.14.19})$$

$$\partial_a \epsilon_{\pm} + \frac{e^z}{2} \Gamma_a \Theta_{\pm} \epsilon_{\pm} = 0 \quad (\text{B.14.20})$$

where

$$\Theta_{\pm} = \Gamma_z + \partial^i A \Gamma_i \mp iAY \Gamma_{z34} - \frac{A}{48} \not{G} C^*, \quad (\text{B.14.21})$$

and, for the z -direction equations, we're considering $r = 0$. The integrability condition for these directions is

$$\Theta_{\mp} \Theta_{\pm} \epsilon_{\pm} = 0, \quad (\text{B.14.22})$$

and we can conclude from this that the solutions to the z -direction equations are

$$\epsilon_{\pm}(0, 0, z, 0, y^i) = \sigma_{\pm}(y^i) + e^{\mp z} \tau_{\pm}(y^i), \quad (\text{B.14.23})$$

where, defining $\Xi_+ = -\frac{1}{2}\Gamma_z \Theta_+$ and $\Xi_- = 1 - \frac{1}{2}\Gamma_z \Theta_-$,

$$\Xi_{\pm} \sigma_{\pm} = 0 \quad (\text{B.14.24})$$

$$\Xi_{\pm} \tau_{\pm} = \mp \tau_{\pm}. \quad (\text{B.14.25})$$

We can write these conditions succinctly as

$$\tilde{\Xi}_{\pm} \chi_{\pm} = 0 \quad (\text{B.14.26})$$

where χ_{\pm} is either σ_{\pm} or τ_{\pm} ,

$$\tilde{\Xi}_{\pm} = \mp \frac{c}{2} - \frac{1}{2} \partial_{\underline{i}} A \Gamma_{zi} \pm \frac{iA}{2} Y \Gamma_{34} - \frac{A}{96} \not{G} \Gamma^z C *, \quad (\text{B.14.27})$$

and c is 1 when $\chi_{\pm} = \sigma_{\pm}$, -1 when $\chi_{\pm} = \tau_{\pm}$.

In the transverse directions, the parallel transport equation is

$$0 = \nabla_{\underline{i}}^{(\pm)} \epsilon_{\pm} \quad (\text{B.14.28})$$

$$= \tilde{\nabla}_{\underline{i}} \epsilon_{\pm} + \Psi_{\underline{i}}^{(\pm)} \epsilon_{\pm}, \quad (\text{B.14.29})$$

where

$$\Psi_{\underline{i}}^{(\pm)} = \pm \frac{1}{2A} \partial_{\underline{i}} A - \frac{i}{2} Q_{\underline{i}} \mp \frac{i}{2} Y \Gamma_{z34i} + \left(-\frac{1}{96} (\Gamma G)_i + \frac{9}{96} \not{G}_i \right) C *. \quad (\text{B.14.30})$$

We can see that this applies independently to σ_{\pm} and τ_{\pm} , so that in general $\nabla_{\underline{i}}^{(\pm)} \chi_{\pm} = 0$.

B.14.5 Maximality Condition on σ_+ and τ_+

We introduce a new operator,

$$\hat{\nabla}_{\underline{i}}^{(+)} = \nabla_{\underline{i}}^{(+)} + \frac{q}{A} \Gamma_{zi} \tilde{\Xi}_+, \quad (\text{B.14.31})$$

with the intention to demonstrate that, for an appropriately chosen value of q , if $\Gamma^i \hat{\nabla}_{\underline{i}}^{(+)} \chi_+ = 0$, then χ_+ satisfies the Killing spinor equations.

As before, we find that

$$\begin{aligned} \tilde{\nabla}^2 \|\chi_+\|^2 &= 2 \left\| \hat{\nabla}^{(+)} \chi_+ \right\|^2 + \frac{1}{2} \tilde{R} \|\chi_+\|^2 \\ &\quad + \text{Re} \left\langle \chi_+, \left[-4 \overline{\Psi}^{(+)\underline{i}} - 2 \Psi^{(+)\underline{i}} - 2 \Gamma^{ij} \Psi_j^{(+)} \right. \right. \\ &\quad \left. \left. - 10 \frac{q}{A} \Gamma^{zi} \tilde{\Xi}_+ \right] \tilde{\nabla}_{\underline{i}} \chi_+ \right\rangle \\ &\quad + \text{Re} \left\langle \chi_+, \left[-2 \left(\overline{\Psi}^{(+)\underline{i}} + \frac{q}{A} \tilde{\Xi}_+ \Gamma^{zi} \right) \left(\Psi_{\underline{i}}^{(+)} + \frac{q}{A} \Gamma_{zi} \tilde{\Xi}_+ \right) \right. \right. \\ &\quad \left. \left. - 2 \tilde{\nabla}^{\underline{i}} \Psi_{\underline{i}}^{(+)} - 2 \Gamma^{ij} \tilde{\nabla}_{\underline{i}} \Psi_j^{(+)} - 10 \tilde{\nabla}_{\underline{i}} \left(\frac{q}{A} \Gamma^{zi} \tilde{\Xi}_+ \right) \right] \chi_+ \right\rangle, \end{aligned} \quad (\text{B.14.32})$$

where

$$\bar{\Psi}_i^{(+)} = \frac{1}{2A} \partial_i A + \frac{i}{2} Q_i + \frac{i}{2} Y \Gamma_{z34i} + \left(-\frac{1}{96} (\Gamma G)_i - \frac{9}{96} \mathcal{G}_i \right) C* \quad (\text{B.14.33})$$

$$\tilde{\Xi}_+ = -\frac{c}{2} + \frac{1}{2} \partial^i A \Gamma_{zi} + \frac{iA}{2} Y \Gamma_{34} - \frac{A}{96} \mathcal{G} \Gamma^z C* . \quad (\text{B.14.34})$$

Expanding the third term, we find that

$$\begin{aligned} & \text{Re} \left\langle \chi_+, \left[-4\bar{\Psi}^{(+)\dot{i}} - 2\Psi^{(+)\dot{i}} - 2\Gamma^{ij} \Psi_j^{(+)} - 10\frac{q}{A} \Gamma^{zi} \tilde{\Xi}_+ \right] \tilde{\nabla}_{\dot{i}} \chi_+ \right\rangle \\ &= \text{Re} \left\langle \chi_+, \left[\frac{5qc}{A} \Gamma^{zi} - \frac{3+5q}{A} \partial^i A - \frac{1+5q}{A} \partial_j A \Gamma^{ij} \right. \right. \\ &\quad \left. \left. - iQ^i + iQ_j \Gamma^{ij} + i(3-5q) Y \Gamma^{z34i} \right. \right. \\ &\quad \left. \left. + \left(\frac{-10+10q}{96} (\Gamma G)^i + \frac{-6+30q}{96} \mathcal{G}^i \right) C* \right] \tilde{\nabla}_{\dot{i}} \chi_+ \right\rangle. \end{aligned} \quad (\text{B.14.35})$$

We would like to write this in the form

$$\alpha \partial^i A \tilde{\nabla}_{\dot{i}} \|\chi_+\|^2 + \text{Re} \left\langle \chi_+, \mathcal{F} \Gamma^i \tilde{\nabla}_{\dot{i}} \chi_+ \right\rangle, \quad (\text{B.14.36})$$

which is only possible if $q = \frac{3}{5}$ in which case

$$\mathcal{F} = \frac{3c}{A} \Gamma^z + \frac{4}{A} \mathcal{G} A - i\mathcal{Q} + \frac{1}{24} \mathcal{G} C* \quad (\text{B.14.37})$$

and $\alpha = -\frac{5}{A}$. We can then further expand the expression as

$$\begin{aligned} & \text{Re} \left\langle \chi_+, \left[-4\bar{\Psi}^{(+)\dot{i}} - 2\Psi^{(+)\dot{i}} - 2\Gamma^{ij} \Psi_j^{(+)} - 10\frac{q}{A} \Gamma^{zi} \tilde{\Xi}_+ \right] \tilde{\nabla}_{\dot{i}} \chi_+ \right\rangle \\ &= -\frac{5}{A} \partial^i A \|\chi_+\|^2 + \text{Re} \left\langle \chi_+, \mathcal{F} \Gamma^i \tilde{\nabla}_{\dot{i}} \chi_+ \right\rangle \end{aligned} \quad (\text{B.14.38})$$

$$= -\frac{5}{A} \partial^i A \|\chi_+\|^2 - \text{Re} \left\langle \chi_+, \mathcal{F} \Gamma^i \left[\Psi_{\dot{i}}^{(+)} + \frac{3}{5A} \Gamma_{zi} \Xi_+ \right] \chi_+ \right\rangle. \quad (\text{B.14.39})$$

Combining this with the bilinear part of the fourth term of equation (B.14.32), we find that

$$\begin{aligned} & \text{Re} \left\langle \chi_+, -2 \left(\bar{\Psi}^{(+)\bar{i}} + \frac{3}{5A} \tilde{\Xi}_+ \Gamma^{zi} + \frac{1}{2} \mathcal{F} \Gamma^i \right) \left(\Psi_i^{(+)} + \frac{3}{5A} \Gamma_{zi} \tilde{\Xi}_+ \right) \chi_+ \right\rangle \\ &= \text{Re} \left\langle \chi_+, -2 \left[\frac{6c}{5A} \Gamma^{zi} + \frac{11}{5A} \partial^i A - \frac{17}{10A} \partial_j A \Gamma^{ij} + \frac{4i}{5} Y \Gamma^{z34i} \right. \right. \end{aligned} \quad (\text{B.14.40})$$

$$\begin{aligned} & \left. + \frac{i}{2} Q_j \Gamma^{ij} + \left(-\frac{1}{40} (\Gamma G)^i - \frac{1}{20} \mathcal{G}^i \right) C^* \right] \\ & \times \left[-\frac{3c}{10A} \Gamma_{zi} + \frac{4}{5A} \partial_i A + \frac{3}{10A} \partial^j A \Gamma_{ij} - \frac{i}{5} Y \Gamma_{z34i} \right. \\ & \left. - \frac{i}{2} Q_i + \left(-\frac{1}{60} (\Gamma G)_i + \frac{3}{40} \mathcal{G}_i \right) C^* \right] \chi_+ \rangle \\ &= \text{Re} \left\langle \chi_+, \left[-\frac{18}{5A^2} - \frac{38}{5A^2} |dA|^2 - \frac{8}{5} Y^2 - \frac{24ic}{5A} Y \Gamma_{34} \right. \right. \end{aligned} \quad (\text{B.14.41})$$

$$\begin{aligned} & \left. - 2 \left(-\frac{1}{40} (\Gamma G)^i - \frac{1}{20} \mathcal{G}^i \right) \left(-\frac{1}{60} \overline{(\Gamma G)}_i + \frac{3}{40} \overline{\mathcal{G}}_i \right) \right. \\ & \left. + \left(\frac{1}{10A} \mathcal{G} \Gamma_z + \frac{1}{16A} \partial_i A (\Gamma G)^i + \frac{i}{12} Q_i (\Gamma G)^i \right) C^* \right] \chi_+ \rangle. \end{aligned}$$

We can use the field equations and Bianci identities to rewrite the last line of (B.14.32),

$$\begin{aligned} & \text{Re} \left\langle \chi_+, \left[-2 \tilde{\nabla}^i \Psi_i^{(+)} - 2 \Gamma^{ij} \tilde{\nabla}_i \Psi_j^{(+)} - \tilde{\nabla}_i \left(\frac{6}{A} \Gamma^{zi} \tilde{\Xi}_+ \right) \right] \chi_+ \right\rangle \\ &= \text{Re} \left\langle \chi_+, \left(\frac{4}{A^2} |dA|^2 - \frac{4}{A} \tilde{\nabla}^2 A + \frac{i}{2} dQ - \frac{1}{48} dG C^* \right) \chi_+ \right\rangle \end{aligned} \quad (\text{B.14.42})$$

$$\begin{aligned} &= \text{Re} \left\langle \chi_+, \left[\frac{16}{A^2} + \frac{20}{A^2} |dA|^2 - 16 Y^2 - \frac{1}{12} |G|^2 + P_i \bar{P}_j \Gamma^{ij} \right. \right. \end{aligned} \quad (\text{B.14.43})$$

$$\left. + \left(-\frac{i}{12} Q_i (\Gamma G)^i + \frac{1}{12} P_i \overline{(\Gamma G)}^i \right) C^* \right] \chi_+ \rangle.$$

The second, third, and fourth terms on the right side of equation (B.14.32) thus sum to

$$\begin{aligned} & \text{Re} \left\langle \chi_+, \left[\frac{12}{5A^2} + \frac{12}{5A^2} |dA|^2 + \frac{12}{5} Y^2 - \frac{24ic}{5A} Y \Gamma_{34} + |P|^2 + P_i \bar{P}_j \Gamma^{ij} \right. \right. \\ & \left. - \frac{1}{40} G_{j_1 j_2}^i \bar{G}_{i k_1 k_2} \Gamma^{j_1 j_2 k_1 k_2} - \frac{1}{80} G_j^{i_1 i_2} \bar{G}_{i_1 j_2 k} \Gamma^{jk} + \frac{1}{60} |G|^2 \right. \\ & \left. + \left(\frac{1}{10A} \mathcal{G} \Gamma_z - \frac{1}{10A} \partial_i A (\Gamma G)^i + \frac{1}{12} P_i \overline{(\Gamma G)}^i \right) C^* \right] \chi_+ \rangle \end{aligned} \quad (\text{B.14.44})$$

Noting that

$$\left\| \tilde{\Xi}_+ \sigma_+ \right\|^2 = \left\langle \chi_+, \tilde{\Xi}_+ \tilde{\Xi}_+ \chi_+ \right\rangle \quad (\text{B.14.45})$$

$$\begin{aligned} &= \left\langle \chi_+, \left[\frac{1}{4} + \frac{1}{4} |dA|^2 + \frac{iA}{2} Y^i \partial_i A \Gamma_{z3} - \frac{iAc}{2} Y \Gamma_{34} + \frac{A^2}{4} Y^2 \right. \right. \\ &\quad \left. \left. - \frac{A^2}{32^2} G^i{}_{j_2 j_3} \bar{G}_{i k_2 k_3} \Gamma^{j_2 j_3 k_2 k_3} + \frac{A^2}{2 \cdot 16^2} G^{i_1 i_2}{}_j \bar{G}_{i_1 i_2 k} \Gamma^{jk} \right. \right. \\ &\quad \left. \left. + \frac{A^2}{16 \cdot 96} |G|^2 + \left(\frac{Ac}{96} \not{G} \Gamma^z - \frac{A}{96} \partial_i A (\Gamma G)^i \right) C_* \right] \chi_+ \right\rangle \end{aligned} \quad (\text{B.14.46})$$

$$\begin{aligned} \|\mathcal{A}\chi_+\|^2 &= \text{Re} \left\langle \chi, \left[|P|^2 + P_i \bar{P}_j \Gamma^{ij} - \frac{1}{64} G^i{}_{j_1 j_2} \bar{G}_{i k_1 k_2} \Gamma^{j_1 j_2 k_1 k_2} \right. \right. \\ &\quad \left. \left. - \frac{1}{32} G^{i_1 i_2}{}_j \bar{G}_{i_1 i_2 k} \Gamma^{jk} + \frac{1}{96} |G|^2 + \frac{1}{12} P_i \overline{(\Gamma G)^i} C_* \right] \chi \right\rangle \end{aligned} \quad (\text{B.14.47})$$

we can now write equation (B.14.32) as

$$\tilde{\nabla}^2 \|\chi\|^2 + \frac{5}{A} \partial^i A \tilde{\nabla}_i \|\chi\|^2 = 2 \left\| \hat{\nabla}^{(+)} \chi \right\|^2 + \frac{48}{5A^2} \|\Xi_c \chi\|^2 + \|\mathcal{A}\chi\|^2 \quad (\text{B.14.48})$$

B.15 IIB AdS₆

B.15.1 Bianchi Identities and Field Equations

We will need to express the field equations and Bianchi identities in terms of the AdS and transverse dimensions. The Einstein equation has both an AdS component and a transverse component,

$$R_{\underline{\mu}\underline{\nu}} = \left[-\frac{5}{A^2} - \frac{5}{A^2} |dA|^2 - \frac{1}{A} \tilde{\nabla}^2 A \right] \eta_{\underline{\mu}\underline{\nu}} = -\frac{1}{48} |G|^2 \eta_{\underline{\mu}\underline{\nu}} \quad (\text{B.15.1})$$

$$\tilde{R}_{ij} = \frac{6}{A} \tilde{\nabla}_i \tilde{\nabla}_j A + \frac{1}{4} G_{(i}{}^{k\ell} \bar{G}_{j)k\ell} - \frac{1}{48} |G|^2 \delta_{ij} + 2P_{(i} \bar{P}_{j)}, \quad (\text{B.15.2})$$

from which we find that

$$\frac{1}{A} \tilde{\nabla}^2 A = \frac{1}{48} |G|^2 - \frac{5}{A^2} - \frac{5}{A^2} |dA|^2 \quad (\text{B.15.3})$$

$$\tilde{R} = \frac{6}{A} \tilde{\nabla}^2 A + \frac{1}{6} |G|^2 + 2|P|^2 \quad (\text{B.15.4})$$

$$= -\frac{30}{A^2} - \frac{30}{A^2} |dA|^2 + \frac{7}{24} |G|^2 + 2|P|^2 \quad (\text{B.15.5})$$

The remaining equations, for $k = 6$, are purely transverse,

$$dP = 2iQ \wedge P \quad (\text{B.15.6})$$

$$dQ = -iP \wedge \bar{P} \quad (\text{B.15.7})$$

$$dG = iQ \wedge G + P \wedge \bar{G} \quad (\text{B.15.8})$$

$$\tilde{\nabla}^i P_i = -\frac{6}{A} \partial^i A P_i + 2iQ^i P_i + \frac{1}{24} G^2 \quad (\text{B.15.9})$$

$$\tilde{\nabla}^k G_{ijk} = -\frac{6}{A} \partial^k A G_{ijk} + iQ^k G_{ijk} + P^k \bar{G}_{ijk} \quad (\text{B.15.10})$$

B.15.2 Differential and Algebraic Killing Spinor Equations on X

The z -direction Killing spinor equations are

$$\partial_z \epsilon_+ = -\frac{1}{2} \Gamma^z \Theta \epsilon_+ \quad (\text{B.15.11})$$

$$\partial_z \epsilon_- = \left(1 - \frac{1}{2} \Gamma^z \Theta\right) \epsilon_- \quad (\text{B.15.12})$$

where $\Theta = \Gamma^z + \partial_i A \Gamma^i - \frac{A}{48} G_{ijk} \Gamma^{ijk} C^*$. Defining $\Xi_+ = -\frac{1}{2} \Gamma^z \Theta$ and $\Xi_- = 1 - \frac{1}{2} \Gamma^z \Theta$, so that,

$$\Xi_{\pm} = \mp \frac{1}{2} + \frac{1}{2} \not{A} \Gamma^z - \frac{A}{96} \not{G} \Gamma^z C^*. \quad (\text{B.15.13})$$

we can write this more succinctly as

$$\partial_z \epsilon_{\pm} = \Xi_{\pm} \epsilon_{\pm}. \quad (\text{B.15.14})$$

It can be shown that the solutions to these equations take the form

$$\epsilon_{\pm}(z, y) = \sigma_{\pm} + e^{\mp z} \tau_{\pm}, \quad (\text{B.15.15})$$

where σ_{\pm} and τ_{\pm} satisfy

$$0 = \Xi_{\pm} \sigma_{\pm} \quad (\text{B.15.16})$$

$$0 = (\Xi_{\pm} \pm 1) \tau_{\pm} \quad (\text{B.15.17})$$

$$= \Xi_{\mp} \tau_{\pm}. \quad (\text{B.15.18})$$

We also have the y -direction Killing spinor equations,

$$0 = \nabla_i^{(\pm)} \epsilon_{\pm} \quad (\text{B.15.19})$$

$$= \tilde{\nabla}_i^{(\pm)} \epsilon_{\pm} + \Psi^{(\pm)} \epsilon_{\pm} \quad (\text{B.15.20})$$

$$\begin{aligned} &= \tilde{\nabla}_i \epsilon_{\pm} + \left(-e^{-z} \frac{r}{A^2} \partial_i A \Gamma_- \Theta \pm \frac{1}{2A} \partial_i A - \frac{i}{2} Q_i \right) \epsilon_{\pm} \\ &\quad + \left(-\frac{1}{96} (\Gamma G)_i + \frac{3}{32} \not{G}_i \right) C^* \epsilon_{\pm}, \end{aligned} \quad (\text{B.15.21})$$

which apply to σ_\pm and τ_\pm independently,

$$\tilde{\nabla}_i \sigma_\pm + \left(\pm \frac{1}{2A} \partial_i A - \frac{i}{2} Q_i \right) \sigma_\pm + \left(-\frac{1}{96} (\Gamma G)_i + \frac{3}{32} \mathcal{G}_i \right) C * \sigma_\pm = 0 \quad (\text{B.15.22})$$

$$\tilde{\nabla}_i \tau_\pm + \left(\pm \frac{1}{2A} \partial_i A - \frac{i}{2} Q_i \right) \tau_\pm + \left(-\frac{1}{96} (\Gamma G)_i + \frac{3}{32} \mathcal{G}_i \right) C * \tau_\pm = 0, \quad (\text{B.15.23})$$

noting that for $r = 0$,

$$\Psi_i^{(\pm)} = \left(\pm \frac{1}{2A} \partial_i A - \frac{i}{2} Q_i \right) + \left(-\frac{1}{96} (\Gamma G)_i + \frac{9}{96} \mathcal{G}_i \right) C * \quad (\text{B.15.24})$$

and the algebraic Killing spinor equation,

$$\overline{P} \epsilon_\pm + \frac{1}{24} \overline{G} C * \epsilon_\pm = 0, \quad (\text{B.15.25})$$

which also applies to σ_\pm and τ_\pm independently. Defining $\mathcal{A} = \overline{P}_i \Gamma^i + \frac{1}{24} \overline{G}_{ijk} \Gamma^{ijk} C *$, this gives

$$\mathcal{A} \sigma_\pm = 0 \quad (\text{B.15.26})$$

$$\mathcal{A} \tau_\pm = 0. \quad (\text{B.15.27})$$

Together, equations (B.15.16), (B.15.18), (B.15.22), (B.15.23), (B.15.26), and (B.15.27) are the Killing spinor equations on the transverse space, X .

B.15.3 Basic Lichnerowicz Theorem

Let ϕ be any spinor. Then we can expand $\tilde{\nabla}^2 \|\phi\|^2$ as

$$\tilde{\nabla}^2 \|\phi\|^2 = 2 \operatorname{Re} \tilde{\nabla}^i \langle \phi, \tilde{\nabla}_i \phi \rangle \quad (\text{B.15.28})$$

$$= 2 \langle \tilde{\nabla}^i \phi, \tilde{\nabla}_i \phi \rangle + 2 \operatorname{Re} \langle \phi, \tilde{\nabla}^i \tilde{\nabla}_i \phi \rangle. \quad (\text{B.15.29})$$

Using the Bianchi identities on the Riemannian curvature tensor, $R_{ij, k\ell}$, we can further expand $\tilde{\nabla}^i \tilde{\nabla}_i \phi$ as

$$\tilde{\nabla}^i \tilde{\nabla}_i \phi = \delta^{ij} \tilde{\nabla}_i \tilde{\nabla}_j \phi \quad (\text{B.15.30})$$

$$= (\Gamma^i \Gamma^j - \Gamma^{ij}) \tilde{\nabla}_i \tilde{\nabla}_j \phi \quad (\text{B.15.31})$$

$$= \Gamma^i \tilde{\nabla}_i (\Gamma^j \tilde{\nabla}_j \phi) - \frac{1}{4} \tilde{R}_{ij, k\ell} \Gamma^{ij} \Gamma^{k\ell} \phi \quad (\text{B.15.32})$$

$$= \Gamma^i \tilde{\nabla}_i (\Gamma^j \tilde{\nabla}_j \phi) + \frac{1}{4} \tilde{R} \phi, \quad (\text{B.15.33})$$

so that

$$\tilde{\nabla}^2 \|\phi\|^2 = 2 \langle \tilde{\nabla}^i \phi, \tilde{\nabla}_i \phi \rangle + 2 \operatorname{Re} \langle \phi, \Gamma^i \tilde{\nabla}_i (\Gamma^j \tilde{\nabla}_j \phi) \rangle - \frac{1}{2} \tilde{R} \langle \phi, \phi \rangle. \quad (\text{B.15.34})$$

B.15.4 Maximality Condition on σ_+ and τ_+

We introduce a new operator, $\hat{\nabla}_i^{(+)}$ defined by

$$\hat{\nabla}_i^{(+)}\chi = \nabla_i^{(+)}\chi + \frac{q}{A}\Gamma_{zi}\Xi_c\chi, \quad (\text{B.15.35})$$

where χ is either σ_+ or τ_+ and $c = 1$ when $\chi = \sigma_+$, -1 when $\chi = \tau_+$. We wish to show that, for an appropriately chosen value of q , if

$$0 = \Gamma^i \hat{\nabla}_i^{(+)}\chi \quad (\text{B.15.36})$$

$$= \Gamma^i \tilde{\nabla}_i\chi + \left[\Gamma^i \Psi_i^{(+)} - 4\frac{q}{A}\Gamma^z\Xi_c \right]\chi \quad (\text{B.15.37})$$

then σ_+ satisfies the Killing spinor equations, (B.15.22), (B.15.16), and (B.15.26). To this end we compute

$$\tilde{\nabla}^2\|\chi\|^2 = 2\langle \tilde{\nabla}^i\chi, \tilde{\nabla}_i\chi \rangle + 2\text{Re}\langle \chi, \Gamma^i \tilde{\nabla}_i(\Gamma^j \tilde{\nabla}_j\chi) \rangle - \frac{1}{2}\tilde{R}\langle \chi, \chi \rangle. \quad (\text{B.15.38})$$

The first term expands to

$$\begin{aligned} 2\langle \tilde{\nabla}^i\chi, \tilde{\nabla}_i\chi \rangle &= 2\|\hat{\nabla}^{(+)}\chi\|^2 - 4\text{Re}\langle (\Psi^{(+)}{}^i + \frac{q}{A}\Gamma^{zi}\Xi_c)\chi, \tilde{\nabla}_i\chi \rangle \\ &\quad - 2\|\left(\Psi^{(+)}{}^i + \frac{q}{A}\Gamma^{zi}\Xi_c\right)\chi\|^2 \\ &= 2\|\hat{\nabla}^{(+)}\chi\|^2 - 4\text{Re}\langle \chi, \left(\bar{\Psi}^{(+)}{}^i - \frac{q}{A}\bar{\Xi}_c\Gamma^{zi}\right)\tilde{\nabla}_i\chi \rangle \end{aligned} \quad (\text{B.15.39})$$

$$\begin{aligned} &\quad - 2\text{Re}\langle \chi, \left(\bar{\Psi}^{(+)}{}^i - \frac{q}{A}\bar{\Xi}_c\Gamma^{zi}\right)(\Psi_i^{(+)} + \frac{q}{A}\Gamma_{zi}\Xi_c)\chi \rangle \\ &= 2\|\hat{\nabla}^{(+)}\chi\|^2 - 4\text{Re}\langle \chi, \bar{\Psi}^{(+)}{}^i \tilde{\nabla}_i\chi \rangle \\ &\quad - 2\text{Re}\langle \chi, \left(\bar{\Psi}^{(+)}{}^i + \frac{q}{A}\bar{\Xi}_c\Gamma^{zi}\right)(\Psi_i^{(+)} + \frac{q}{A}\Gamma_{zi}\Xi_c)\chi \rangle \end{aligned} \quad (\text{B.15.40})$$

where

$$\bar{\Psi}_i^{(+)} = \frac{1}{2A}\partial_i A + \frac{i}{2}Q_i + \left(-\frac{1}{96}(\Gamma G)_i - \frac{9}{96}\mathcal{G}_i\right)C^* \quad (\text{B.15.41})$$

$$\bar{\Xi}_c = -\frac{c}{2} + \frac{1}{2}\partial_i A \Gamma^{zi} - \frac{A}{96}\mathcal{G}\Gamma^z C^* \quad (\text{B.15.42})$$

The second term expands to

$$\begin{aligned} &2\text{Re}\langle \chi, \Gamma^i \tilde{\nabla}_i(\Gamma^j \tilde{\nabla}_j\chi) \rangle \\ &= -2\text{Re}\langle \chi, \Gamma^i \tilde{\nabla}_i \left[\Gamma^j \Psi_j^{(+)}\chi - 4\frac{q}{A}\Gamma^z\Xi_c\chi \right] \rangle \end{aligned} \quad (\text{B.15.43})$$

$$\begin{aligned} &= -2\text{Re}\langle \chi, \Gamma^i \left[\Gamma^j \Psi_j^{(+)} - 4\frac{q}{A}\Gamma^z\Xi_c \right] \tilde{\nabla}_i\chi \\ &\quad + \left[\Gamma^i \Gamma^j \tilde{\nabla}_i \Psi_j^{(+)} - 4\left(\frac{q}{A^2} \tilde{\nabla}_i A \Gamma^{zi} \Xi_c - \frac{q}{A} \Gamma^{zi} \tilde{\nabla}_i \Xi_c \right) \right] \chi \rangle. \end{aligned} \quad (\text{B.15.44})$$

Combining these, we can rewrite equation (B.15.38) as

$$\begin{aligned}\tilde{\nabla}^2 \|\chi\|^2 &= 2 \left\| \hat{\nabla}^{(+)} \chi \right\|^2 + \frac{1}{2} \tilde{R} \|\chi\|^2 \\ &+ \text{Re} \left\langle \chi, \left[-4\bar{\Psi}^{(+i)} - 2\Psi^{(+i)} - 2\Gamma^{ij}\Psi_j^{(+)} - 8\frac{q}{A}\Gamma^{zi}\Xi_c \right] \tilde{\nabla}_i \chi \right\rangle \\ &+ \text{Re} \left\langle \chi, \left[-2\left(\bar{\Psi}^{(+i)} + \frac{q}{A}\Xi_c\Gamma^{zi}\right) \left(\Psi_i^{(+)} + \frac{q}{A}\Gamma_{zi}\Xi_c\right) \right. \right. \\ &\quad \left. \left. - 2\tilde{\nabla}^i \Psi_i^{(+)} - 2\Gamma^{ij}\tilde{\nabla}_i \Psi_j^{(+)} - 8\tilde{\nabla}_i \left(\frac{q}{A}\Gamma^{zi}\Xi_c\right) \right] \chi \right\rangle.\end{aligned}\tag{B.15.45}$$

Using the fact that $\text{Re}\langle \phi, \Gamma^{ij}\phi \rangle = \text{Re}\langle \phi, \Gamma^{ij}C * \phi \rangle = 0$, we can expand the third term,

$$\begin{aligned}&\text{Re} \left\langle \chi, \left[-4\bar{\Psi}^{(+i)} - 2\Psi^{(+i)} - 2\Gamma^{ij}\Psi_j^{(+)} - 8\frac{q}{A}\Gamma^{zi}\Xi_c \right] \tilde{\nabla}_i \chi \right\rangle \\ &= \text{Re} \left\langle \chi, \left[\frac{4cq}{A}\Gamma^{zi} - \frac{3+4q}{A}\partial^i A - \frac{1+4q}{A}\partial_j A \Gamma^{ij} - iQ^i + iQ_j \Gamma^{ij} \right. \right. \\ &\quad \left. \left. + \left(\frac{-12+8q}{96}(\Gamma G)^i + \frac{-12+24q}{96}G^i \right) C * \right] \tilde{\nabla}_i \chi \right\rangle.\end{aligned}\tag{B.15.46}$$

We want to choose q such that this can be written as

$$\alpha \partial^i A \tilde{\nabla}_i \|\chi\|^2 + \text{Re} \left\langle \chi, \mathcal{F} \Gamma^i \tilde{\nabla}_i \chi \right\rangle,\tag{B.15.47}$$

which is only possible if $q = 1$. Thus,

$$\begin{aligned}&\text{Re} \left\langle \chi, \left[-4\bar{\Psi}^{(+i)} - 2\Psi^{(+i)} - 2\Gamma^{ij}\Psi_j^{(+)} - 8\frac{q}{A}\Gamma^{zi}\Xi_c \right] \tilde{\nabla}_i \chi \right\rangle \\ &= \text{Re} \left\langle \chi, \left[\frac{4c}{A}\Gamma^{zi} - \frac{7}{A}\partial^i A - \frac{5}{A}\partial_j A \Gamma^{ij} - iQ^i + iQ_j \Gamma^{ij} \right. \right. \\ &\quad \left. \left. + \left(-\frac{1}{24}(\Gamma G)^i + \frac{1}{8}G^i \right) C * \right] \tilde{\nabla}_i \chi \right\rangle\end{aligned}\tag{B.15.48}$$

$$= -\frac{6}{A} \partial^i A \tilde{\nabla}_i \|\chi\|^2 + \text{Re} \left\langle \chi, \left[\frac{4c}{A}\Gamma^z + \frac{5}{A}\partial_j A \Gamma^j - iQ_j \Gamma^j + \frac{1}{24}G C * \right] \Gamma^i \tilde{\nabla}_i \chi \right\rangle\tag{B.15.49}$$

$$= -\frac{6}{A} \partial^i A \tilde{\nabla}_i \|\chi\|^2 - \text{Re} \left\langle \chi, \mathcal{F} \left[\Gamma^i \Psi_i^{(+)} - \frac{4}{A}\Gamma^z \Xi_c \right] \chi \right\rangle\tag{B.15.50}$$

$$= -\frac{6}{A} \partial^i A \tilde{\nabla}_i \|\chi\|^2 - \text{Re} \left\langle \chi, \mathcal{F} \Gamma^i \left[\Psi_i^{(+)} + \frac{1}{A}\Gamma_{zi}\Xi_c \right] \chi \right\rangle\tag{B.15.51}$$

where

$$\mathcal{F} = \frac{4c}{A}\Gamma^z + \frac{5}{A}\partial A - iQ + \frac{1}{24}G C *.\tag{B.15.52}$$

Combining this with the bilinear part of the fourth term in (B.15.45), we find that

$$\begin{aligned} & \text{Re} \left\langle \chi, -2 \left(\bar{\Psi}^{(+i)} + \frac{1}{A} \bar{\Xi}_c \Gamma^{zi} + \frac{1}{2} \mathcal{F} \Gamma^i \right) \left(\Psi_i^{(+)} + \frac{1}{A} \Gamma_{zi} \Xi_c \right) \chi \right\rangle \\ &= \text{Re} \left\langle \chi, -2 \left[\frac{3c}{2A} \Gamma^{zi} + \frac{5}{2A} \partial^i A - \frac{2}{A} \partial_j A \Gamma^{ij} + \frac{i}{2} Q_j \Gamma^{ij} + \left(-\frac{2}{96} (\Gamma G)^i - \frac{6}{96} \mathcal{G}^i \right) C^* \right] \right. \\ & \quad \times \left. \left[-\frac{c}{2A} \Gamma_{zi} + \frac{1}{A} \partial_i A + \frac{1}{2A} \partial^j A \Gamma_{ij} - \frac{i}{2} Q_i + \left(-\frac{2}{96} (\Gamma G)_i + \frac{6}{96} \mathcal{G}_i \right) C^* \right] \chi \right\rangle \quad (\text{B.15.53}) \end{aligned}$$

$$\begin{aligned} &= \text{Re} \left\langle \chi, \left[-\frac{6}{A^2} - \frac{11}{A^2} |dA|^2 - 2 \left(-\frac{2}{96} (\Gamma G)^i - \frac{6}{96} \mathcal{G}^i \right) \left(-\frac{2}{96} \overline{(\Gamma G)}_i + \frac{6}{96} \overline{\mathcal{G}}_i \right) \right. \right. \\ & \quad \left. \left. + \left(\frac{16c}{96A} \mathcal{G} \Gamma^z - \frac{16}{96A} \partial_i A (\Gamma G)^i + \frac{i}{12} Q_i (\Gamma G)^i \right) C^* \right] \chi \right\rangle \quad (\text{B.15.54}) \end{aligned}$$

We can also use the Bianchi identities and field equations to expand the derivatives in the fourth term on the right side of equation (B.15.45),

$$\begin{aligned} & \text{Re} \left\langle \chi, \left[-2 \tilde{\nabla}^i \Psi_i^{(+)} - 2 \Gamma^{ij} \tilde{\nabla}_i \Psi_j^{(+)} - 8 \tilde{\nabla}_i \left(\frac{1}{A} \Gamma^{zi} \Xi_c \right) \right] \chi \right\rangle \\ &= \text{Re} \left\langle \chi, \left[\frac{5}{A^2} |dA|^2 - \frac{5}{A} \tilde{\nabla}^2 A + \frac{i}{2} dQ - \frac{1}{48} dG C^* \right] \chi \right\rangle \quad (\text{B.15.55}) \end{aligned}$$

$$\begin{aligned} &= \text{Re} \left\langle \chi, \left[\frac{25}{A^2} + \frac{30}{A^2} |dA|^2 - \frac{5}{48} |G|^2 + P_i \bar{P}_j \Gamma^{ij} \right. \right. \\ & \quad \left. \left. + \left(-\frac{i}{12} Q_i (\Gamma G)^i + \frac{1}{12} P_i \overline{(\Gamma G)}^i \right) C^* \right] \chi \right\rangle. \quad (\text{B.15.56}) \end{aligned}$$

The second, third, and fourth terms on the right side of equation (B.15.45) thus sum to

$$\begin{aligned} & \text{Re} \left\langle \chi, \left[\frac{4}{A^2} + \frac{4}{A^2} |dA|^2 + \frac{1}{48} |G|^2 + |P|^2 + P_i \bar{P}_j \Gamma^{ij} \right. \right. \\ & \quad \left. \left. + \left(\frac{16c}{96A} \mathcal{G} \Gamma^z - \frac{6}{96A} \partial_i A (\Gamma G)^i + \frac{1}{12} P_i \overline{(\Gamma G)}^i \right) C^* \right] \chi \right\rangle \quad (\text{B.15.57}) \end{aligned}$$

Noting that

$$\|\Xi_c \chi\|^2 = \text{Re} \langle \chi, \bar{\Xi}_c \Xi_c \chi \rangle \quad (\text{B.15.58})$$

$$\begin{aligned} &= \text{Re} \left\langle \chi, \left[\frac{1}{4} + \frac{1}{4} |dA|^2 + \frac{3A^2}{16 \cdot 96} G^{ij}{}_k \bar{G}_{ij\ell} \Gamma^{k\ell} + \frac{A^2}{16 \cdot 96} |G|^2 \right. \right. \\ & \quad \left. \left. + \left(\frac{Ac}{96} \mathcal{G} \Gamma^z - \frac{A}{96} \partial_i A (\Gamma G)^i \right) C^* \right] \chi \right\rangle \quad (\text{B.15.59}) \end{aligned}$$

$$\|\mathcal{A} \chi\|^2 = \text{Re} \langle \chi, \bar{\mathcal{A}} \mathcal{A} \chi \rangle \quad (\text{B.15.60})$$

$$= \text{Re} \left\langle \chi, \left[|P|^2 + P_i \bar{P}_j \Gamma^{ij} - \frac{3}{96} G^{ij}{}_k \bar{G}_{ij\ell} \Gamma^{k\ell} + \frac{1}{96} |G|^2 + \frac{1}{12} P_i \overline{(\Gamma G)}^i C^* \right] \chi \right\rangle \quad (\text{B.15.61})$$

we can now write equation (B.15.45) as

$$\tilde{\nabla}^2 \|\chi\|^2 + \frac{6}{A} \partial^i A \tilde{\nabla}_i \|\chi\|^2 = 2 \|\hat{\nabla}^{(+)} \chi\|^2 + \frac{16}{A^2} \|\Xi_c \chi\|^2 + \|\mathcal{A} \chi\|^2 \quad (\text{B.15.62})$$

Bibliography

- [1] M. Ammon and J. Erdmenger. *Gauge/Gravity Duality: Foundations and Applications*. Cambridge University Press, 2015.
- [2] B. Zwiebach. *A First Course in String Theory*. Cambridge University Press, 2009.
- [3] Peter G. O. Freund and Mark A. Rubin. Dynamics of Dimensional Reduction. *Phys. Lett.*, B97:233–235, 1980.
- [4] M. J. Duff, B. E. W. Nilsson, and C. N. Pope. Kaluza-Klein Supergravity. *Phys. Rept.*, 130:1–142, 1986.
- [5] E. Cremmer, B. Julia, and Joel Scherk. Supergravity Theory in Eleven-Dimensions. *Phys. Lett.*, B76:409–412, 1978.
- [6] Juan Martin Maldacena. The Large N limit of superconformal field theories and supergravity. *Int. J. Theor. Phys.*, 38:1113–1133, 1999. [Adv. Theor. Math. Phys.2,231(1998)].
- [7] Ofer Aharony, Steven S. Gubser, Juan Martin Maldacena, Hirosi Ooguri, and Yaron Oz. Large N field theories, string theory and gravity. *Phys. Rept.*, 323:183–386, 2000.
- [8] U. Gran, J. Gutowski, U. Kayani, and G. Papadopoulos. Dynamical symmetry enhancement near IIA horizons. *JHEP*, 06:139, 2015.
- [9] U. Gran, J. Gutowski, U. Kayani, and G. Papadopoulos. Dynamical symmetry enhancement near massive IIA horizons. *Class. Quant. Grav.*, 32(23):235004, 2015.
- [10] U. Gran, J. Gutowski, and G. Papadopoulos. IIB horizons. *Class. Quant. Grav.*, 30:205004, 2013.
- [11] U. Gran, J. Gutowski, and G. Papadopoulos. Index theory and dynamical symmetry enhancement near IIB horizons. *JHEP*, 11:104, 2013.
- [12] J. Gutowski and G. Papadopoulos. Index theory and dynamical symmetry enhancement of M-horizons. *JHEP*, 05:088, 2013.

- [13] J. B. Gutowski and G. Papadopoulos. Supersymmetry of AdS and flat backgrounds in M-theory. *JHEP*, 02:145, 2015.
- [14] S. W. Beck, J. B. Gutowski, and G. Papadopoulos. Supersymmetry of AdS and flat IIB backgrounds. *JHEP*, 02:020, 2015.
- [15] S. Beck, J. B. Gutowski, and G. Papadopoulos. Supersymmetry of IIA warped flux AdS and flat backgrounds. *JHEP*, 09:135, 2015.
- [16] S. W. Beck, J. B. Gutowski, and G. Papadopoulos. Geometry and supersymmetry of heterotic warped flux AdS backgrounds. *JHEP*, 07:152, 2015.
- [17] S. W. Beck, J. B. Gutowski, and G. Papadopoulos. AdS₅ Backgrounds with 24 Supersymmetries. 2016.
- [18] Juan Maldacena. The Gauge/gravity duality. 2011.
- [19] U. Gran, J. Gutowski, and G. Papadopoulos. AdS backgrounds from black hole horizons. *Class. Quant. Grav.*, 30:055014, 2013.
- [20] J. Gutowski and G. Papadopoulos. Heterotic Black Horizons. *JHEP*, 07:011, 2010.
- [21] J. Gutowski and G. Papadopoulos. Heterotic horizons, Monge-Ampere equation and del Pezzo surfaces. *JHEP*, 10:084, 2010.
- [22] J. Grover, J. B. Gutowski, G. Papadopoulos, and W. A. Sabra. Index Theory and Supersymmetry of 5D Horizons. *JHEP*, 06:020, 2014.
- [23] Igor R. Klebanov and Edward Witten. Superconformal field theory on three-branes at a Calabi-Yau singularity. *Nucl. Phys.*, B536:199–218, 1998.
- [24] Niall T. Macpherson, Carlos Núñez, Leopoldo A. Pando Zayas, Vincent G. J. Rodgers, and Catherine A. Whiting. Type IIB supergravity solutions with AdS₅ from Abelian and non-Abelian T dualities. *JHEP*, 02:040, 2015.
- [25] L. J. Romans. New Compactifications of Chiral $N = 2d = 10$ Supergravity. *Phys. Lett.*, B153:392, 1985.
- [26] Mirjam Cvetič, Hong Lu, C. N. Pope, and Justin F. Vazquez-Poritz. AdS in warped spacetimes. *Phys. Rev.*, D62:122003, 2000.
- [27] Bobby Samir Acharya, J. M. Figueroa-O’Farrill, C. M. Hull, and Bill J. Spence. Branes at conical singularities and holography. *Adv. Theor. Math. Phys.*, 2:1249–1286, 1999.
- [28] Jerome P. Gauntlett, Dario Martelli, James Sparks, and Daniel Waldram. Supersymmetric AdS(5) solutions of type IIB supergravity. *Class. Quant. Grav.*, 23:4693–4718, 2006.

- [29] Fabio Apruzzi, Marco Fazzi, Achilleas Passias, Andrea Rota, and Alessandro Tomasiello. Six-Dimensional Superconformal Theories and their Compactifications from Type IIA Supergravity. *Phys. Rev. Lett.*, 115(6):061601, 2015.
- [30] Dieter Lust and Dimitrios Tsimpis. New supersymmetric AdS(4) type II vacua. *JHEP*, 09:098, 2009.
- [31] Jerome P. Gauntlett, Nakwoo Kim, and Daniel Waldram. Supersymmetric AdS(3), AdS(2) and Bubble Solutions. *JHEP*, 04:005, 2007.
- [32] Mariana Grana. Flux compactifications in string theory: A Comprehensive review. *Phys. Rept.*, 423:91–158, 2006.
- [33] John H. Schwarz and Peter C. West. Symmetries and Transformations of Chiral N=2 D=10 Supergravity. *Phys. Lett.*, B126:301, 1983.
- [34] John H. Schwarz. Covariant Field Equations of Chiral N=2 D=10 Supergravity. *Nucl. Phys.*, B226:269, 1983.
- [35] U. Gran, J. Gutowski, G. Papadopoulos, and D. Roest. N=31 is not IIB. *JHEP*, 02:044, 2007.
- [36] U. Gran, J. Gutowski, G. Papadopoulos, and D. Roest. IIB solutions with N>28 Killing spinors are maximally supersymmetric. *JHEP*, 12:070, 2007.
- [37] Jose M. Figueroa-O’Farrill and George Papadopoulos. Maximally supersymmetric solutions of ten-dimensional and eleven-dimensional supergravities. *JHEP*, 03:048, 2003.
- [38] U. Gran, J. Gutowski, and G. Papadopoulos. Classification of IIB backgrounds with 28 supersymmetries. *JHEP*, 01:044, 2010.
- [39] Jose Figueroa-O’Farrill and Noel Hustler. The homogeneity theorem for supergravity backgrounds. *JHEP*, 10:014, 2012.
- [40] Juan Martin Maldacena and Carlos Nunez. Supergravity description of field theories on curved manifolds and a no go theorem. *Int. J. Mod. Phys.*, A16:822–855, 2001. [,182(2000)].
- [41] Fabio Apruzzi, Marco Fazzi, Achilleas Passias, Dario Rosa, and Alessandro Tomasiello. AdS₆ solutions of type II supergravity. *JHEP*, 11:099, 2014. [Erratum: JHEP05,012(2015)].
- [42] Dieter Lust and Dimitrios Tsimpis. Supersymmetric AdS(4) compactifications of IIA supergravity. *JHEP*, 02:027, 2005.
- [43] G. W. Gibbons and P. K. Townsend. Vacuum interpolation in supergravity via super p-branes. *Phys. Rev. Lett.*, 71:3754–3757, 1993.

- [44] Fabio Apruzzi, Marco Fazzi, Dario Rosa, and Alessandro Tomasiello. All AdS_7 solutions of type II supergravity. *JHEP*, 04:064, 2014.
- [45] L. J. Romans. Massive $\text{N}=2$ Supergravity in Ten-Dimensions. *Phys. Lett.*, B169:374, 1986.
- [46] I. C. G. Campbell and Peter C. West. $\text{N}=2$ $\text{D}=10$ Nonchiral Supergravity and Its Spontaneous Compactification. *Nucl. Phys.*, B243:112, 1984.
- [47] E. A. Bergshoeff, J. Hartong, P. S. Howe, T. Ortin, and F. Riccioni. IIA/IIB Supergravity and Ten-forms. *JHEP*, 05:061, 2010.
- [48] Ulf Gran, George Papadopoulos, and Christian von Schultz. Supersymmetric geometries of IIA supergravity I. *JHEP*, 05:024, 2014.
- [49] Igor A. Bandos, Jose A. de Azcarraga, and Oscar Varela. On the absence of BPS preonic solutions in IIA and IIB supergravities. *JHEP*, 09:009, 2006.
- [50] Fabio Apruzzi, Marco Fazzi, Achilleas Passias, and Alessandro Tomasiello. Supersymmetric AdS_5 solutions of massive IIA supergravity. *JHEP*, 06:195, 2015.
- [51] Maxime Gabella, Dario Martelli, Achilleas Passias, and James Sparks. $\mathcal{N} = 2$ supersymmetric AdS_4 solutions of M-theory. *Commun. Math. Phys.*, 325:487–525, 2014.
- [52] L. Castellani, L. J. Romans, and N. P. Warner. A Classification of Compactifying Solutions for $d = 11$ Supergravity. *Nucl. Phys.*, B241:429, 1984.
- [53] C. N. Pope and N. P. Warner. Two New Classes of Compactifications of $d = 11$ Supergravity. *Class. Quant. Grav.*, 2:L1, 1985.
- [54] M. Huq and M. A. Namazie. Kaluza-Klein Supergravity in Ten-dimensions. *Class. Quant. Grav.*, 2:293, 1985. [Erratum: *Class. Quant. Grav.* 2,597(1985)].
- [55] F. Giani and M. Pernici. $\text{N}=2$ SUPERGRAVITY IN TEN-DIMENSIONS. *Phys. Rev.*, D30:325–333, 1984.
- [56] Stephen R. Green, Emil J. Martinec, Callum Quigley, and Savdeep Sethi. Constraints on String Cosmology. *Class. Quant. Grav.*, 29:075006, 2012.
- [57] Fridrik Freyr Gautason, Daniel Junghans, and Marco Zagermann. On Cosmological Constants from α' -Corrections. *JHEP*, 06:029, 2012.
- [58] M.B. Green, J.H. Schwarz, and E. Witten. *Superstring Theory: Volumes 1 and 2*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1988.
- [59] A. Opfermann and G. Papadopoulos. Homogeneous HKT and QKT manifolds. 1998.
- [60] Paul S. Howe and G. Papadopoulos. Twistor spaces for HKT manifolds. *Phys. Lett.*, B379:80–86, 1996.

- [61] S. Ivanov and G. Papadopoulos. A No go theorem for string warped compactifications. *Phys. Lett.*, B497:309–316, 2001.
- [62] S. Ivanov and G. Papadopoulos. Vanishing theorems and string backgrounds. *Class. Quant. Grav.*, 18:1089–1110, 2001.
- [63] C. M. Hull. Compactifications of the Heterotic Superstring. *Phys. Lett.*, B178:357, 1986.
- [64] A. Strominger and Edward Witten. New Manifolds for Superstring Compactification. *Commun. Math. Phys.*, 101:341, 1985.
- [65] Katrin Becker, Melanie Becker, Ji-Xiang Fu, Li-Sheng Tseng, and Shing-Tung Yau. Anomaly cancellation and smooth non-Kähler solutions in heterotic string theory. *Nucl. Phys.*, B751:108–128, 2006.
- [66] Ji-Xiang Fu and Shing-Tung Yau. The Theory of superstring with flux on non-Kähler manifolds and the complex Monge-Ampère equation. *J. Diff. Geom.*, 78(3):369–428, 2008.
- [67] Marisa Fernandez, Stefan Ivanov, Luis Ugarte, and Raquel Villacampa. Non-Kähler Heterotic String Compactifications with non-zero fluxes and constant dilaton. *Commun. Math. Phys.*, 288:677–697, 2009.
- [68] U. Gran, P. Lohrmann, and G. Papadopoulos. The Spinorial geometry of supersymmetric heterotic string backgrounds. *JHEP*, 02:063, 2006.
- [69] Thomas Friedrich and Stefan Ivanov. Killing spinor equations in dimension 7 and geometry of integrable $G(2)$ manifolds. *J. Geom. Phys.*, 48:1, 2003.
- [70] George Papadopoulos. New half supersymmetric solutions of the heterotic string. *Class. Quant. Grav.*, 26:135001, 2009.
- [71] J. Gillard, G. Papadopoulos, and D. Tsimpis. Anomaly, fluxes and $(2,0)$ heterotic string compactifications. *JHEP*, 06:035, 2003.
- [72] C. M. Hull and P. K. Townsend. The Two Loop Beta Function for σ Models With Torsion. *Phys. Lett.*, B191:115, 1987.
- [73] E. A. Bergshoeff and M. de Roo. The Quartic Effective Action of the Heterotic String and Supersymmetry. *Nucl. Phys.*, B328:439, 1989.
- [74] Paul S. Howe and G. Papadopoulos. Anomalies in Two-dimensional Supersymmetric Non-linear σ Models. *Class. Quant. Grav.*, 4:1749–1766, 1987.
- [75] Paul S. Howe and G. Papadopoulos. Finiteness and anomalies in $(4,0)$ supersymmetric sigma models. *Nucl. Phys.*, B381:360–372, 1992.

- [76] Dario Martelli and James Sparks. Non-Kahler heterotic rotations. *Adv. Theor. Math. Phys.*, 15(1):131–174, 2011.
- [77] Paul S. Howe and G. Papadopoulos. Ultraviolet Behavior of Two-dimensional Supersymmetric Nonlinear σ Models. *Nucl. Phys.*, B289:264–276, 1987.
- [78] Jerome P. Gauntlett, Dario Martelli, James Sparks, and Daniel Waldram. Supersymmetric AdS(5) solutions of M theory. *Class. Quant. Grav.*, 21:4335–4366, 2004.
- [79] Juan Martin Maldacena and Carlos Nunez. Supergravity description of field theories on curved manifolds and a no go theorem. *Int. J. Mod. Phys.*, A16:822–855, 2001. [,182(2000)].
- [80] Davide Gaiotto and Juan Maldacena. The Gravity duals of N=2 superconformal field theories. *JHEP*, 10:189, 2012.
- [81] Georgios Itsios, Carlos Nunez, Konstadinos Sfetsos, and Daniel C. Thompson. On Non-Abelian T-Duality and new N=1 backgrounds. *Phys. Lett.*, B721:342–346, 2013.
- [82] Georgios Itsios, Carlos Nunez, Konstadinos Sfetsos, and Daniel C. Thompson. Non-Abelian T-duality and the AdS/CFT correspondence:new N=1 backgrounds. *Nucl. Phys.*, B873:1–64, 2013.
- [83] Krzysztof Pilch and Nicholas P. Warner. A New supersymmetric compactification of chiral IIB supergravity. *Phys. Lett.*, B487:22–29, 2000.
- [84] Ibrahima Bah, Christopher Beem, Nikolay Bobev, and Brian Wecht. AdS/CFT Dual Pairs from M5-Branes on Riemann Surfaces. *Phys. Rev.*, D85:121901, 2012.
- [85] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky, and E. Sokatchev. Unconstrained Off-Shell N=3 Supersymmetric Yang-Mills Theory. *Class. Quant. Grav.*, 2:155, 1985.
- [86] F. Delduc and J. McCabe. The Quantization of $N = 3$ Superyang-mills Off-shell in Harmonic Superspace. *Class. Quant. Grav.*, 6:233, 1989.
- [87] Ofer Aharony and Mikhail Evtikhiev. On four dimensional N=3 superconformal theories. 2015.
- [88] Iñaki García-Etxebarria and Diego Regalado. N=3 four dimensional field theories. 2015.
- [89] J. B. Gutowski A. Fontanella and G. Papadopoulos. page to appear.
- [90] Andrew Strominger. AdS(2) quantum gravity and string theory. *JHEP*, 01:007, 1999.
- [91] Ulf Gran, George Papadopoulos, and Christian von Schultz. Supersymmetric geometries of IIA supergravity II. *JHEP*, 12:113, 2015.
- [92] U. Gran, J. Gutowski, G. Papadopoulos, and D. Roest. Systematics of IIB spinorial geometry. *Class. Quant. Grav.*, 23:1617–1678, 2006.